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RECURSIVE FORMULAS OF HYPEREDGE DECOMPOSITION SETS

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ABSTRACT The concepts of the undirected and directed decompositions are introduced for a hyperedge. Then, the recursive formulas of the underected decomposition set SD(m) and directed decomposition set SPD(m) are derived for an m-vertex hyperedge. Furthermore, the recursive formulas of their cardinalities |SD(m)| and |SPD(m)| are yielded.
KEY WORDS active hypernetwork, directed hypergraph theory, hyperedge undirected decomposition, hyperedge directed decomposition.

AN active hypernetwork N is the interconnection of $e \geqq 2$ multiterminal active and passive networks called subnetworks. Its topological representation is a directed hypergraph $H=(V,E)$ consisting of a terminal set V and a hyperedge set E, in which directed and undirected hyperedges are corresponding to the active and passive subnetworks in N respectively. Refs.[1] and [2] present the fundamentals of directed hypergraph theory. Ref.[3] gives a survey of the development of directed hypergraph theory and its some applications on electrical network analysis and synthesis. The undirected decomposition set SD(m) and directed decomposition set SPD(m) of an m-terminal hyperedge are of great use in many applications of directed hypergraph theory. For example, in Ref.[4] , by applying SD(m) , an exact decomposition algorithm is proposed for finding network overall reliability; and in Ref.[5], by applying SD(m) and SPD(m) , the DCP, PDCP and GDC methods are presented for analysing hypernetworks, whichare improvements and developments to Chen's ECP method. To find SD(m) and SPD(m) efficiently, the recursive formulas presented in this paper can be employed.

1 RECURSIVE FORMULA OF HYPEREDGE UNDIRECTED DECOMPOSITION SET SD(m)

An undirected h-decomposition $D=D(m,h)$ of an m-terminal (undirect or directed) hyperedge $F=F[1\cdots m]=\{1,\cdots,m\}$ is a partition of F into h disjoint nonempty subsets $F_1,\cdots,F_h$, that is:

$$D=F[F_1,\cdots,F_h]=\{F_1,\cdots,F_h\},\ F=\cup_{i=1}^{h}F_i, F_i\cap F_j=\phi,\ i\neq j \tag{1}$$

where F_i is called the i-th undirected subhyperedge of F with its terminal juxtaposition as its simplified expression. For example, $F=F[123]=\{1,2,3\}$ has an undirected

2-decomposition $F[12,3]=\{12,3\}$. The number of h-decompositions of an m-terminal hyperedge is denoted by $[m,h]$, called the stirling number of the second kind[6]. The set of all undirected decompositions of F is called the undirected decomposition set of F, and denoted by SD(m).

Theorem 1 (Recursive Formula of SD(m)) Let the undirected decomposition set of an m-terminal(undirected or directed) hyperedge $F[1\cdots m]=\{1,\cdots,m\}$ be

$$\mathrm{SD}(m) = U_{h=1}^{m}\mathrm{SD}(m,h) \qquad (2-1)$$

$$\mathrm{SD}(m,h) = \{D(m,h,a)|a = 1,\cdots,[m,h]\} \qquad (2-2)$$

$$D(m,h,a) = \{F[m,h,a,b]|b = 1,\cdots,h\} \qquad (2-3)$$

Where SD(m,h) and $D(m,h,a)$ are the undirected h-decomposition set and the a-th undirected h-decomposition of $F[1\cdots m]$, respectively, and $F(m,h,a,b)$ is the b-th undirected subhyperedge of $D(m,h,a)$. Then the undirected decomposition set of (m+1)-terminal hyperedge $F[1\cdots(m+1)]=\{1,\cdots,m+1\}$ is

$$\mathrm{SD}(m+1)=U_{h=1}^{m+1}\mathrm{SD}(m+1,h) \qquad (3-1)$$

$$\mathrm{SD}(m+1,h) = \{D(m+1,h,a)|a = 1,\cdots,[m+1,h]\} \qquad (3-2)$$

$$D(m+1,h,a) = \begin{cases} \{F[m,h-1,a,1],\cdots,F[m,h-1,a,h-1],m+1\}, \\ \quad a = 1,\cdots,[m,h-1]; \\ \{F[m,h,d,1],\cdots,F[m,h,d,b](m+1),\cdots,F[m,h,d,h]\}, \\ \quad d = 1,\cdots,[m,h];\ b = 1,\cdots,h;\ a = [m,h-1]+(d-1)h+b. \end{cases} \qquad (3-3)$$

and the cardinality of SD(m+1,h) is

$$|\mathrm{SD}(m+1,h)| = [m+1,h] = [m,h-1] + h[m,h] \qquad (4)$$

Proof Formulas (3 − 1) and (3 − 2) are obvious by definitions. Let us prove formula (3 − 3). Either adding one subhyperedge (m+1) into each (h − 1)-decompisition $D(m,h-1,a)$ of $F[1\cdots m]$ or adding one terminal (m+1) into each subhyperedge $F[m,h,d,b]$ of every h-decomposition $D(m,h,d)$ can yield one h-decomposition $D(m+1,h,a)$ of $F[1\cdots(m+1)]$. This will yield all h-decompositions of $F[1\cdots(m+1)]$, hence formula (3 − 3) follows. Formula (4) can be derived easily from formula (3 − 3).

Corollary 1 By using formula (3), starting from SD(1)={{1}}, we can obtain

SD(2)={{1 2},{1,2}}
SD(3)={{1 2 3},{12,3},{13,2},{1,23},{1,2,3}}
SD(4)={{1 2 3 4},{123,4},{124,3},{12,34},{134,2},{13,24},{14,23} , { 1,234},
{12,3,4},{13,2,4},{1,23,4},{14,2,3},{1,24,3},{1,2,34},{1,2,3,4}}
⋮

Obviously, to obtain the undirected decomposition set $\mathrm{SD}(m|F[i_1\cdots i_m])$ of any m-terminal hyperedge $F[i_1\cdots i_m]=\{i_1,\cdots,i_m\}$, we have need only to carry out the following terminal label transformation for SD(m), SD(m,h) and $D(m,h,a)$ in Theorem 1 and Corrollary 1: $1\cdots m \rightarrow i_1\cdots i_m$(i.e., $1\rightarrow i_1,\cdots, m\rightarrow i_m$), denoted by $\mathrm{SD}(m|F[i_1\cdots i_m]) = \mathrm{SD}(m)[1\cdots m \rightarrow i_1\cdots i_m]$.

Corollary 2 The total number of undirected decompositions of an m-terminal hyper-

-edge $F[1\cdots m]$ is

$$|\mathrm{SD}(m)| = B_m = [m,1] + \cdots + [m,m] \quad (5)$$

where $[m,h], h=1,\cdots,m$, can be obtained by using recursive formula (4), B_m is called the Bell number[6]. Starting from $B_1=1$, by applying formulas (4) and (5), we can obtain

$B_1=1$

$B_2=1+1=2$

$B_3=1+3+1=5$

$B_4=1+7+6+1=15$

$B_5=1+15+25+10+1=52$

$\vdots$

Reordering the elements $D(m,2,a)$ of $SD(m,2)$ in increasing order of cardinalities $n = |F[\mathrm{m}, 2,\mathrm{a},1]$ of their first subhyperedges, and adding in a 2-decomposition $D_2(m,m,1)=\{1\cdots m, \Phi\}$ (Φ is an empty set), the resulting set is called an increasing ordering 2-decomposition set, and denoted by $\mathrm{SD}_2(m)$ as follows

$$\mathrm{SD}_2(m)=U_{n=1}^{m}\mathrm{SD}_2(m,n) \quad (6-1)$$

$$\mathrm{SD}_2(m,n) = \{D_2(m,n,a)|a = 1,\cdots,C_{m-1}^{n-1}\} \quad (6-2)$$

$$D_2(m,n,a) = \{F_1[m,n,a],F_2[m,n,a]\},|F_1[m,n,a]| = n \quad (6-3)$$

where C_{m-1}^{n-1} is the combinational number of selecting $n-1$ terminals from m-1 terminals. Since $F_1[m,n,a]$ must contain terminal 1, from other $m-1$ terminals selecting out any n − 1 terminals and adding in terminal 1 can form one $F_1[m,n,a]$ and one 2-decomposition $D_2(m,n,a)$, hence $|\mathrm{SD}_2(m,n)| = C_{m-1}^{n-1}$. Thus we have

$$|\mathrm{SD}_2(m)| = \sum_{n=1}^{m} C_{m-1}^{n-1} = 2^{m-1} \quad (7)$$

Corollary 3 From Corollary 1 and formula (6) we can obtain

$\mathrm{SD}_2(1)=\{1,\Phi\}$

$\mathrm{SD}_2(2)=\{\{1,2\},\{12,\Phi\}\}$

$\mathrm{SD}_2(3)=\{\{1,23\},\{12,3\},\{13,2\},\{123,\Phi\}\}$

$\mathrm{SD}_2(4)=\{\{1,234\},\{12,34\},\{13,24\},\{14,23\},\{123,4\},\{124,3\},\{134,2\},\{1234,\Phi\}\}$

$\vdots$

Corollary 3 will be used to derive the recursive formula of SPD(m) inthe following section.

2 RECURSIVE FORMULA OF HYPEREDGE DIRECTED DECOMPOSITION SET SPD(m)

A positive-rooted directed h-decomposition $\mathrm{PD}(m,h)_r$, with a positive root set $r=\{r_1,\cdots, r_n\}$, of an m-terminal directed hyperedge $F(1\cdots m)=\{1,\cdots,m\}$ is a partition of $F(1\cdots m)$

into h nonempty positive-rooted subhyperedges $r_i w_i$ with a positive root r_i (w_i is a negative pole set or empty set), $i=1,\cdots,h$, and denoted by

$$\mathrm{PD}(m,h)_r=F(rw)=F(r_1w_1,\cdots,r_hw_h)=\{r_1w_1,\cdots,r_hw_h\} \tag{8}$$

where $rw=r_1w_1,\cdots,r_hw_h$. The positive-rooted directed decomposition set $\mathrm{SPD}(m)_p$, with a given positive-root set $p=\{p_1,\cdots,p_n\}$, of $F(1\cdots m)$ is the set of all its positive-rooted directed decompositions $\mathrm{PD}(m,h,c)_r$ with a root set $r\supseteq p$, that is,

$$\mathrm{SPD}(m)_p=\{\mathrm{PD}(m,h,c)_r|r\supseteq p;\ h=|p|,\cdots,m;\ c=1,\cdots,(m,h)_p\} \tag{9}$$

where $r=\{r_1,\cdots,r_h\}=\{p_1,\cdots,p_n,r_{n+1},\cdots,r_n\}$, and $(m,h)_p$ is the number of all positive-rooted directed h-decompositions $\mathrm{PD}(m,h,c)_r$ with a given positive-root set $p\subseteq r$ of $F(1\cdots m)$. For example, $\mathrm{SPD}(3)_{2,3}=\{\{21,3\},\{2,31\},\{2,3,1\}\}$, $\mathrm{SPD}(3)_1=\{\{123\},\{12,3\},\{13,2\},\{1,23\},\{1,32\},\{1,2,3\}\}$. The usual case is given $p=1$. For simplicity, let $\mathrm{SPD}(m)=\mathrm{SPD}(m)_1$, $(m,h)=(m,h)_1$, and $\mathrm{PD}(m,h,c)=\mathrm{PD}(m,h,c)_r(r\supseteq 1)$. Obviously, to obtain the positive-rooted directed h-decomposition $\mathrm{PD}(m,h,c|F(i_1\cdots i_m))$ and postive-rooted directed decomsition set $\mathrm{SPD}(m|F(i_1\cdots i_m))$ of any m-terminal hyperedge $F(i_1\cdots i_m)=\{i_1,\cdots,i_m\}$, we have need only to carry out the terminal label transformation $1\cdots m\rightarrow i_1\cdots i_m$ for $\mathrm{PD}(m,h,c)$ and $\mathrm{SPD}(m)$, denoted by $\mathrm{PD}(m,h,c|F(i_1\cdots i_m))=\mathrm{PD}(m,h,c)(1\cdots m\rightarrow i_1\cdots i_m)$, $\mathrm{SPD}(m|F(i_1\cdots i_m))=\mathrm{SPD}(m)(1\cdots m\rightarrow i_1\cdots i_m)$.

Theorem 2 (Recursive Formula of SPD(m)) Let the directed decomposition set with positive root 1 of an m-terminal directed hyperedge $F(1\cdots m)=\{1,\cdots,m\}$ be

$$\mathrm{SPD}(m)=U_{h=1}^{m}\mathrm{SPD}(m,h) \tag{10-1}$$

$$\mathrm{SPD}(m,h)=\{\mathrm{PD}(m,h,c)|c=1,\cdots,(m,h)\} \tag{10-2}$$

$$\mathrm{PD}(m,h,c)=\{F(m,h,c,b)|b=1,\cdots,h\} \tag{10-3}$$

where $\mathrm{SPD}(m,h)$ and $\mathrm{PD}(m,h,c)$ are the positive-rooted directed h-decomposition set and the c-th positive-rooted h-decomposition of $F(1\cdots m)$, respectively, and $F(m,h,c,b)$ is the b-th positive-rooted subhyperedge of $\mathrm{PD}(m,h,c)$. let $C_0^0=1$, $(m,0)=0$, $(m,h|h>m)=0$. Then the positive-rooted directed decomposition set of $(m+1)$-terminal directed hyperedge $F(1\cdots(m+1))=\{1,\cdots,m+1\}$ is

$$\mathrm{SPD}(m+1)=U_{h=1}^{m+1}\mathrm{SPD}(m+1,h) \tag{11-1}$$

$$\mathrm{SPD}(m+1,h)=\{\mathrm{PD}(m+1,h,c)|c=1,\cdots,(m+1,h)\} \tag{11-2}$$

$$\mathrm{PD}(m+1,h,c)=\begin{cases}\{F(m,h,d,1),\cdots,F(m,h,d,b)(m+1),\cdots,F(m,h,d,h)\}\\ \quad d=1,\cdots,(m,h);\ b=1,\cdots,h;\ c=(d-1)h+b;\\ \{F_1[m,n,a]\}\cup\mathrm{PD}(m-n+1,h-1,d|(m+1)F_2[m,n,a]),\\ \quad n=1,\cdots,m-h+2;\ a=1,\cdots,C_{m-1}^{n-1};\\ \quad d=1,\cdots,(m-n+1,h-1);\\ c=h(m,h)+\sum_{i=1}^{m-h+2}u(n-i)C_{m-1}^{i-1}(m-i+1,h-1)+(a-1)\\ \quad (m-n+1,h-1)+d.\end{cases} \tag{11-3}$$

where $F_1[m,n,a]$ and $F_2[m,n,a]$ are the first and second subhyperedges of increasing order-ing 2-decompostion $D_2(m,n,a)$, $(m,h) = |\mathrm{SPD}(m,h)|$, $u(n-i)$ is a step function, i.e.,

$$u(n-i) = \begin{cases} 1, & n > i \\ 0, & n \le i \end{cases} \tag{12}$$

Furthermore, the cardinality of SPD(m+1,h) is

$$|\mathrm{SPD}(m+1,h)| = (m+1,h) = h(m,h) + \sum_{n=1}^{m-h+2} C_{m-1}^{n-1}(m-n+1,h-1) \tag{13}$$

Proof Formulas (11 − 1) and (11 − 2) are obvious by definitions. Let us prove formula (11 − 3). Adding one negative-pole terminal (m+1) into each positive rooted subhyperedge $F(m,h,d,b)$ of every positive-rooted h-decomposition PD(m,h,d) can yield one positive-rooted h-decompositionof $F(1 \cdots (m+1))$. For all possible values of d and b, we can obtain all directed h-decompositions with positive root 1 and negative pole (m+1) of $F(1 \cdots (m+1))$. $F_1[m,n,a]$ contains terminal 1, here we consider it as one subhyperedge with positive root 1. Hyperedge $(m+1)F_2[m,n,a]$ contains (m − n+1) terminals. Its each directed (h − 1)-decomposition with positive root (m+1) is

$$\mathrm{PD}(m-n+1,h-1,d|(m+1)F_2[m,n,a]) = \mathrm{PD}(m-n+1,h-1,d)(1 \cdots (m-n+1) \rightarrow (m+1)F_2(m,n,a)) \tag{14}$$

which contains h − 1 directed subhyperedges. The union of this set and set $\{F_1(m,n,a)\}$ is a direcrted h-decomposition containing positive root set {1,m+1}. For all possible values of n, a and d, we can obtain all directed h-decompositions containing positive root set {1,m+1} of $F(1 \cdots (m+1))$. The above two forms yield all directed h-decompositions containing positive root 1 of $F(1 \cdots (m+1))$. Hence formula (11 − 3) follows. Let $n = m - h+2$, $a=C_{m-1}^{n-1}$, $d=(m-n+1, h-1)$ in the expression of c in formula (11 − 3) , formula (13) can be derived.

Corollary 4 By applying recursive formula (11), starting from SPD(1)={{1}}, we can obtain

SPD(1)={{1}}
SPD(2)={{12},{1,2}}
SPD(3)={{123},{13,2},{1,23},{1,32},{12,3},{1,3,2}}
SPD(4)={{1234},{134,2},{13,24},{14,23},{1,234},{14,32},{1,324},{124,3},
{12,34},{1,423},{12,43},{13,42},{123,4},{14,3, 2},{1,34,2},{1,3,24},
{1,43,2},{1,4,23},{1,42,3},{12,4,3},{13,4,2},{1,4,3,2}}
⋮

Corollary 5 The total number of all directed decompositions of an m-terminal directed hyperedge $F(1 \cdots m)$, called m-terminal directed decomposition number and denoted by H_m, is

$$|\mathrm{SPD}(m)| = H_m = (m,1) + \cdots + (m,m) \tag{15}$$

where (m,h), h=1,⋯,m, can be derived by recursive formula (13) . Starting from H_1=1, by applying formulas (13) and (15) we can obtain

H_1=1

H_2=1+1=2

H_3=1+4+1=6

H_4=1+12+9+1=23

H_5=1+32+54+16+1=104

⋮

For example, $H_4=1(3,1)+[2(3,2)+C_2^0(3,1)+C_2^1(2,1)+C_2^2(1,1)]+[3(3,3)+C_2^0(3,2)+C_2^1(2,2)]+C_2^0(3,3)=1+12+9+1=23$

The correctness of Corollarys 4 and 5 can be verified by applying the Corollary 1.3 (Algorithm for finging SPD$(m)_p$) and Corollary 1.4 in Ref.[2].

3 CONCLUSIONS

Theorem 2 and Corollaries 4 and 5 are new recursive formulas for positive-rooted directed decomposition set SPD(m) and m-terminal directed decomposition number H_m. They expose two explicit relations among several SPD(m)'s and H_m's with some different values of m, respectively, and can be used to generate SPD(m) and H_m recursively.

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超边分解集的递推公式

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摘要 引入了超边的无向分解与有向分解的概念，导出了 m 点超边的无向分解集 SD(m)和有向分解集 SPD(m)的递推公式，进而得到它们的基数 |SD(m)| 和 |SPD(m)| 递推公式.

关键词 有源超网络，有向超图埋论，超边无向分解，超边有向分解