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Retrospective and prospective review of the generalized nonlinear strength theory for geomaterials

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Abstract: Strength theory is the basic theory for calculating and designing the strength of engineering materials in civil, hydraulic, mechanical, aerospace, military, and other engineering disciplines. Therefore, the comprehensive study of the generalized nonlinear strength theory (GNST) of geomaterials has significance for the construction of engineering rock strength. This paper reviews the GNST of geomaterials to demonstrate the research status of nonlinear strength characteristics of geomaterials under complex stress paths. First, it systematically summarizes the research progress of GNST (classical and empirical criteria). Then, the latest research the authors conducted over the past five years on the GNST is introduced, and a generalized three-dimensional (3D) nonlinear Hoek–Brown (HB) criterion (NGHB criterion) is proposed for practical applications. This criterion can be degenerated into the existing three modified HB criteria and has a better prediction performance. The strength prediction errors for six rocks and two *in-situ* rock masses are 2.0724%–3.5091% and 1.0144%–3.2321%, respectively. Finally, the development and outlook of the GNST are expounded, and a new topic about the building strength index of rock mass and determining the strength of *in-situ* engineering rock mass is proposed. The summarization of the GNST provides theoretical traceability and optimization for constructing *in-situ* engineering rock mass strength.

Keywords: rock mechanics; rock mass strength; strength theory; failure criterion; Hoek–Brown criterion; intermediate principal stress; deviatoric plane; smoothness and convexity

1. Introduction

Strength theory is the science of studying the regularities of material yielding or failure under complex stress states. Generally, the failure (strength) criterion is adopted to judge the material yield and failure, which characterizes the relationship between the stress state and strength parameters of the material under the limit state (failure condition) [1–2]. The generalized failure includes the material transformation from an elastic to a plastic state; in other words, strength theory comprises the yield and failure criteria. Strength theory (or criterion) is an interdisciplinary field studied by physicists, material scientists, geoscientists, mechanical engineers, civil engineers, and mining engineers.

In the past 30 years, with the successful solution of complex geological and rock engineering problems in many projects, the development of rock mechanics has been strongly promoted. These projects include the Qinghai–Tibet railway, central Yunnan water diversion, south-to-north water diversion, west-to-east power transmission, west-to-east gas transmission, the national traffic trunk lines, resource exploitation, urban underground space development and use, high and steep slopes, and deep buried long large tunnels. However, ensuring the economy of engineering construction, reducing the high carbonization of the engineering construction lifecycle [3], and avoiding rock engineering disasters always involve the *in-situ* rock mass strength, which has been the core bottleneck concerning underground engineering. This primary research is about the source of the underground engineering system and is directly related to the national economic operation and disaster prevention and control. Therefore, it is of considerable theoretical value to summarize and prospect the research progress of the generalized nonlinear strength theory (GNST) for building a new generation of *in-situ* engineering rock mass strength under complex stress conditions. It has critical strategic significance for developing national underground engineering.

Over 100 failure criteria have been proposed, and tens of thousands of application research papers on failure criteria have been published [4–5]. Based on the differences in these established theoretical frameworks, the failure criteria can be divided into classical and empirical (e.g., the HB criterion). Classical criteria, such as Mohr–Coulomb (MC) and Drucker–Prager (DP), have been widely used in geotechnical en-

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gineering. However, the classical criteria have many limitations. Some criteria only apply to specific geomaterials or can be fitted according to the test results of particular geomaterials and lack a unified understanding of the fundamental strength characteristics of geomaterials. Not all mathematical forms and parameter meanings of criteria with the same theoretical basis are the same, and some parameters even have unclear physical meanings. For nonlinear empirical criteria, a theoretical system with the HB criterion as the core has been formed through continuous development and improvement. The HB criterion can be applied to rock and rock mass, and the material constants can be obtained by laboratory experiments and mineral composition and discontinuity descriptions [6]. However, the HB criterion does not consider the influence of intermediate principal stress (IPS), making it challenging to determine its parameters accurately. Thus, it has poor applicability to jointed rocks with apparent anisotropy.

Therefore, many scholars have extended the single classical and empirical criteria and proposed the GNST [7], such as the single- or multiparameter three-shear strength theory [8], the MC series strength theory [9], the Matsuoka–Nakai (MN) series strength theory [10], HB series strength theory [11], and other strength theories [12]. "Generalized" is synonymous with "unified," indicating that the criterion applies to various materials. "Nonlinear" indicates the feature of the criterion, i.e., the failure envelope on the deviatoric or meridian planes is a continuous and smooth curve. The "strength theory" includes a yield criterion describing plastic deformation and a failure criterion representing failure conditions. GNST (classical and empirical criteria) research is primarily based on laboratory experiment results or theoretical derivation.

Since its establishment, the International Society for Rock Mechanics and Rock Engineering (ISRM) has defined its work objective: determining "what is rock mass strength." After over 60 years of development, many scholars have made considerable efforts and achieved remarkable results in large-scale slopes, long large tunnels, ground foundations under complex geological conditions, hydropower dams, and energy exploitation [13–15]. However, the core problem of *in-situ* rock mass strength has not been effectively solved.

This paper systematically describes the research progress of the GNST, summarizes and condenses the advantages and disadvantages of various GNSTs, and graphicalizes most GNSTs, which play a positive guiding role in further understanding them. Based on the defects of the existing GNSTs, the authors present the latest research on generalized nonlinear classical and HB criteria. A generalized 3D nonlinear HB (NGHB) criterion is proposed, and research on rock and in-situ rock mass strength prediction is performed. The NGHB criterion lays a theoretical foundation for establishing *in-situ* engineering rock mass strength under complex stress conditions.

2. Research status of generalized nonlinear classical criteria

Over the past 100 years, many scholars have devoted themselves to studying the strength characteristics of geomaterials and established numerous generalized nonlinear classical failure criteria (classical GNFCs). This goal can be achieved using the following standard steps. (1) Establish the GNFC based on single or multiple classical criteria. (2) Propose a new classical criterion improved by the deviatoric plane function (also called the shape function [16]). (3) Improve the classical criteria based on laboratory experiments. (4) Improve the classical criteria by considering the anisotropy of geomaterials. (5) Improve the classical criteria using other methods, such as the IPS coefficient. This paper reviews the research on classical GNFCs from the above five aspects.

The GNSTs (classical and empirical criteria) described in this paper are frequently expressed in terms of principal stresses or stress invariants. To understand and clarify the relationship between the two representations, it is first necessary to elucidate their relevant concepts.

2.1. Stress state in terms of stress invariants

It is assumed that the geomaterials are isotropic, and the compressive stress is specified to be positive. Fig. 1 [17] expresses the stress state at any point in the principal stress space as $P(\sigma_1, \sigma_2, \sigma_3)$, where σ_1, σ_2, and σ_3 represent the three principal stresses. Rays with equal inclination angles to the

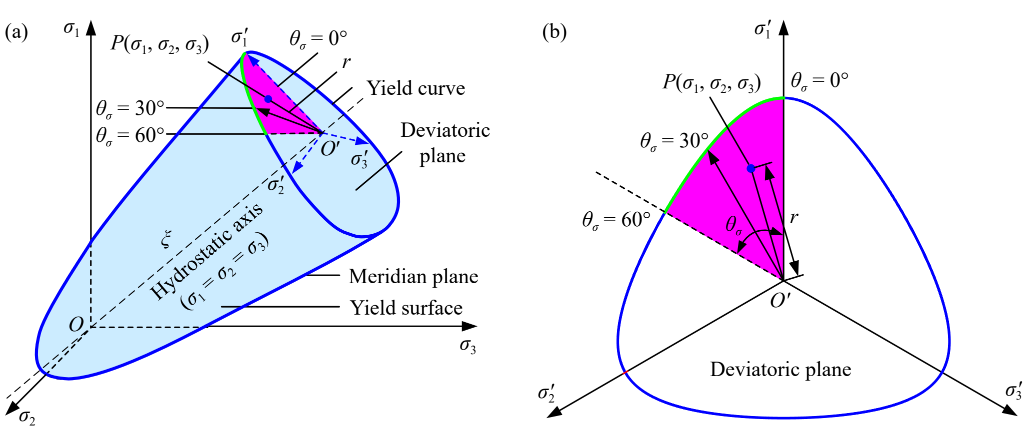

Fig. 1. 3D failure (yield) surface and deviatoric plane in the principal stress space [17]: (a) space coordinate system; (b) deviatoric plane coordinate system (σ_1', σ_2', and σ_3' are the projections of the principal stresses σ_1, σ_2, and σ_3 on the deviatoric plane, respectively).

three coordinate axes in the principal stress space are called space diagonals (the hydrostatic axis). The plane where the hydrostatic axis lies is the meridian plane, and the plane orthogonal to the meridian plane is the deviatoric plane [1]. The deviatoric plane passing through point O is the π plane. The stress state at any point on the deviatoric plane is typically described by the π plane normal stress ξ or mean stress p, π plane shear stress r or generalized shear stress q, and Lode angle θ_σ or θ. ξ represents the distance from the deviatoric plane to the origin O, whereas r is that from the stress point P to the hydrostatic axis. The Lode angle θ_σ ($0 \le \theta_\sigma \le \pi/3$) is the parameter describing the position of the stress point P on the deviatoric plane. According to Nayak and Zienkiewicz [18], the invariants (ξ, r, θ_σ) or (p, q, θ_σ) are defined as

$$\begin{cases} \xi = \dfrac{I_1}{\sqrt{3}} \text{ or } p = \dfrac{I_1}{3} \\ r = \sqrt{2J_2} \text{ or } q = \sqrt{3J_2} \\ \theta_\sigma = \dfrac{1}{3}\cos^{-1}\left(\dfrac{3\sqrt{3}}{2}\dfrac{J_3}{J_2^{3/2}}\right) \end{cases} \quad (1)$$

where I_1 is the first invariant of the stress tensor, and J_2 and J_3 are the second and third invariants of the stress deviator tensor, respectively.

Based on the invariants I_1, J_2, and θ_σ, the principal stresses σ_1, σ_2, and σ_3 can be expressed as

$$\begin{cases} \sigma_1 = \dfrac{1}{3}I_1 + \dfrac{2}{\sqrt{3}}\sqrt{J_2}\cos\theta_\sigma \\ \sigma_2 = \dfrac{1}{3}I_1 + \dfrac{2}{\sqrt{3}}\sqrt{J_2}\cos\left(\theta_\sigma - \dfrac{2\pi}{3}\right) \\ \sigma_3 = \dfrac{1}{3}I_1 + \dfrac{2}{\sqrt{3}}\sqrt{J_2}\cos\left(\theta_\sigma + \dfrac{2\pi}{3}\right) \end{cases}, \quad 0 \le \theta_\sigma \le \dfrac{\pi}{3} \quad (2)$$

2.2. General curved yield criteria

2.2.1. General yield criteria between the Tresca and von Mises criteria

Hershey [19], Davis [20], Hosford [21], Barlat and Lian [22], Tan [23], Karafillis and Boyce [24], and Owen and Perić [25] proposed general curved yield criteria, containing a range of yield criteria between the Tresca and von Mises criteria. Karafillis and Boyce [24] expressed a general curved yield criterion between the Tresca and von Mises criteria as

$$\Phi = (\sigma_1 - \sigma_2)^{2k} + (\sigma_2 - \sigma_3)^{2k} + (\sigma_3 - \sigma_1)^{2k} = 2\sigma_0^{2k} \quad (3)$$

where k is the material constant ranging in [1, +∞), and σ_0 is the yield stress in uniaxial tension.

Eq. (3) is related to the stress yield function of Davis [20] and is a general form of Bialey's flow law [26]. Its yield surface is inside the von Mises yield criterion and outside the Tresca yield criterion. This criterion (Eq. (3)) is also called the Bailey–Davis criterion.

2.2.2. General yield criteria between the twin–shear and von Mises (or Tresca) criteria

Karafillis and Boyce [24] proposed two general curved yield criteria. One is a series between the von Mises and twin–shear yield criteria (Fig. 2(a) [24]), and the other is those between the Tresca and twin–shear yield criteria (Fig. 2(b) [24]).

2.2.3. Edelman–Drueker and Hosford criteria

Edelman and Drucker [27] proposed the criterion named after them, expressed as

$$F = J_2^3 - C_d J_3^2 \quad (4)$$

where C_d is the material constant and F is the yield function.

Dodd and Naruse [28] extended the Edelman–Drueker criterion and proposed the Dodd–Naruse criterion 1, expressed as

$$F = \left(J_2^3\right)^m - C_d\left(J_3^2\right)^m \quad (5)$$

where m is the material constant.

When the material constant m is an integer, the Dodd–Naruse criterion 1 exhibits equal yield stress under tensile and compressive conditions, and if $\sigma_3 = 0$, it is expressed as

$$\frac{\sigma_1}{\sigma_u} = \left\{\frac{\left(\dfrac{1}{3}\right)^{3m} - C_d\left(\dfrac{2}{27}\right)^{2m}}{\left[\dfrac{1}{3}(1-\varepsilon+\varepsilon^2)\right]^{3m} - C_d\left[\dfrac{1}{27}(2-\varepsilon)(1-2\varepsilon)(1+\varepsilon)\right]^{2m}}\right\}^{\frac{1}{6}} \quad (6)$$

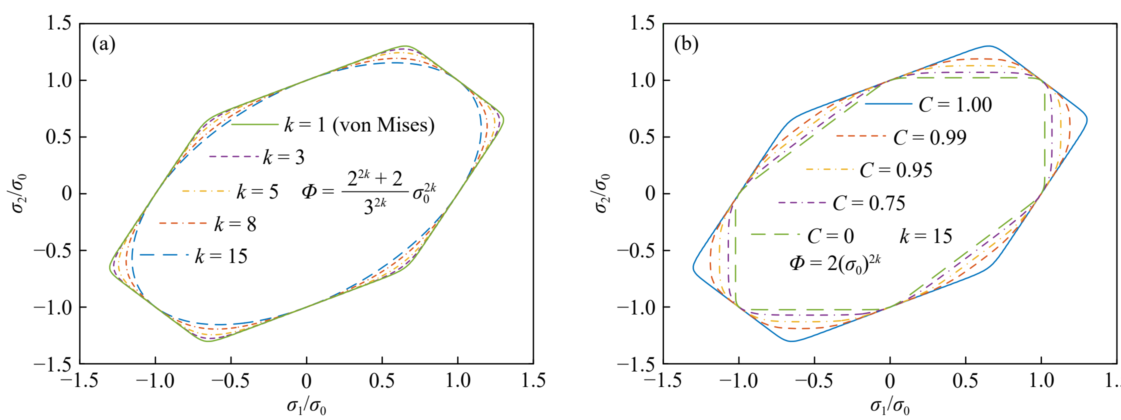

Fig. 2. Failure envelope characteristics of the general curved yield criteria on the σ_1–σ_2 plane [24]: (a) curved general yield criterion between von Mises and twin–shear yield criteria; (b) curved general yield criteria between Tresca and twin–shear yield criteria (C is the mixing factor). Reprinted from *J. Mech. Phys. Solids.*, 41, A.P. Karafillis and M.C. Boyce, A general anisotropic yield criterion using bounds and a transformation weighting tensor, 1859-1886, Copyright 1994, with permission from Elsevier.

$$\varepsilon = \frac{\sigma_2}{\sigma_1} \tag{7}$$

where σ_u is the uniaxial yield stress. The material constant ε ranges in (−∞, 0.5] and the material constant C_d in [−27/8, 9/4].

Later, Dodd and Naruse [28] proposed a generalized yield criterion (the Dodd–Naruse criterion 2), expressed as

$$\begin{cases} \left[\frac{1}{2}(\sigma_1-\sigma_2)^t + \frac{1}{2}(\sigma_1-\sigma_3)^t\right]^{\frac{1}{t}} = \sigma_u, \\ (\sigma_1-\sigma_2) \geq (\sigma_2-\sigma_3) \\ \left[\frac{1}{2}(\sigma_1-\sigma_3)^t + \frac{1}{2}(\sigma_2-\sigma_3)^t\right]^{\frac{1}{t}} = \sigma_u, \\ (\sigma_1-\sigma_2) < (\sigma_2-\sigma_3) \end{cases} \tag{8}$$

where t is a material constant that takes values in the range [1, +∞).

When t approaches positive infinity, the Dodd–Naruse criterion 2 degenerates to the Tresca criterion. When $t \in$ [1, +∞), Eq. (8) can give a series of yield curves, all of which have corner points (Fig. 3(a) [28]).

Furthermore, Hosford [21] proposed a straightforward general form of isotropic yield criterion (Hosford criterion), which is closer to the experimental and theoretical results than the von Mises or Tresca criteria. It is expressed as

$$\left[\frac{(\sigma_1-\sigma_2)^t + (\sigma_2-\sigma_3)^t + (\sigma_1-\sigma_3)^t}{2}\right]^{\frac{1}{t}} = \sigma_0 \tag{9}$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3,\ 1 \leq t \leq +\infty \tag{10}$$

When $t = 2$, the Hosford criterion degenerates to the von Mises criterion, and when $t = 1$, this criterion degenerates to the Tresca criterion (Fig. 3(b) [21]).

2.2.4. Simplified anisotropic yield criteria

Hill [29] proposed a new yield criterion, expressed as

$$\begin{aligned} &f'|\sigma_2-\sigma_3|^w + g'|\sigma_3-\sigma_1|^w + h'|\sigma_1-\sigma_2|^w + \\ &a'|2\sigma_1-\sigma_2-\sigma_3|^w + b'|2\sigma_2-\sigma_1-\sigma_3|^w + \\ &c'|2\sigma_3-\sigma_1-\sigma_2|^w = \sigma_s^w \end{aligned} \tag{11}$$

where σ_s is a scaling factor for the stresses, w is the material constant ($w \geq 1$), and f', g', h', a', b', and c' are constants reflecting the anisotropic characteristics. When $f' = g' = h'$ and $a' = b' = c'$, the isotropic case becomes a three-parameter criterion.

Dodd and Naruse [28] improved Eq. (11) ($f' = g' = h' = 1$), obtaining the following expression:

$$\begin{aligned} &|\sigma_2-\sigma_3|^w + |\sigma_3-\sigma_1|^w + |\sigma_1-\sigma_2|^w + a'|2\sigma_1-\sigma_2-\sigma_3|^w + \\ &b'|2\sigma_2-\sigma_1-\sigma_3|^w + c'|2\sigma_3-\sigma_1-\sigma_2|^w = \sigma_s^w \end{aligned} \tag{12}$$

When $w \in$ [1, +∞), a range of curved yield criteria is obtained between the single-shear and twin-shear stress yield criteria.

Hosford [21] and Barlat *et al.* [22,30–31] also introduced similar anisotropic yield criteria.

2.2.5. Other generalized nonlinear failure criteria

Zienkiewicz [32] proposed a general expression of the unified yield criteria, which can attribute most yield criteria to the theoretical framework of the general formula. It is still the basis on which many scholars propose the unified yield criteria. Later, Desai [33], de Boer [34], Shen [35], and Krenk [36] proposed new unified yield criteria in polynomial form.

The general curved yield criteria mentioned in Section 2.2 satisfy the smoothness and convexity requirements. Most are between the Tresca and von Mises criteria or the single- and twin-shear yield criteria. These criteria are inconvenient for elastic-plastic analysis [37].

2.3. GNFCs capable of degenerating into a single criterion

Aubertin *et al.* [38] proposed the von Mises and DP unified (MSDPu) criterion, the failure envelope of which the deviatoric plane degenerates into a circle, as described by the von Mises or DP criteria. Zhou *et al.* [39] proposed a revised DP yield criterion by referring to the modified DP criterion of the deviatoric plane function of the MSDPu criterion. Zhou and Li [40] proposed a GNFC with a DP criterion expression based on the experimental study of friction materials. This criterion can degenerate into the DP criterion, which can describe the strength characteristics of various materials from the MC criterion (exclusive) to the DP criterion (Fig. 4).

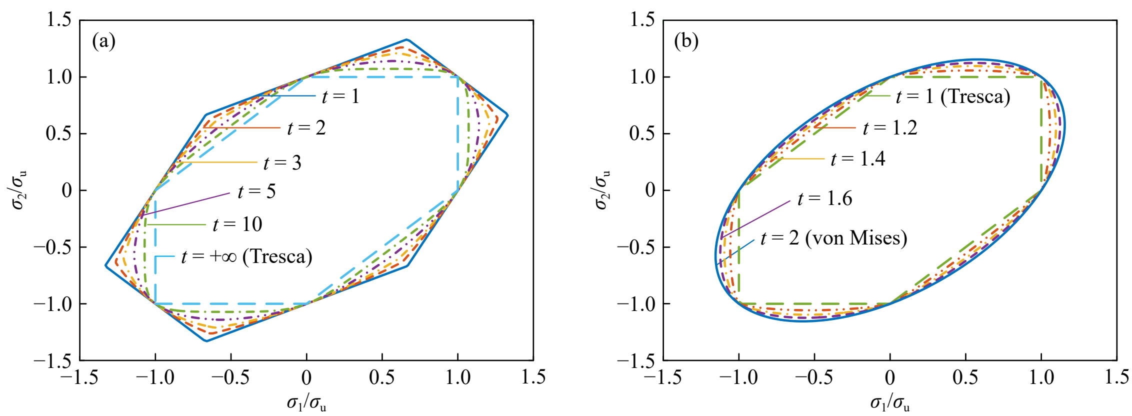

Fig. 3. Failure envelope characteristics of (a) the Dodd–Naruse criterion 2 [28] and (b) the Hosford criterion [21] on the σ_1–σ_2 plane. (a) Reprinted from *Int. J. Mech. Sci.*, 31, B. Dodd, K. Naruse, Limitations on isotropic yield criteria, 511-519, Copyright 1989, with permission from Elsevier. (b) Adapted from [21].

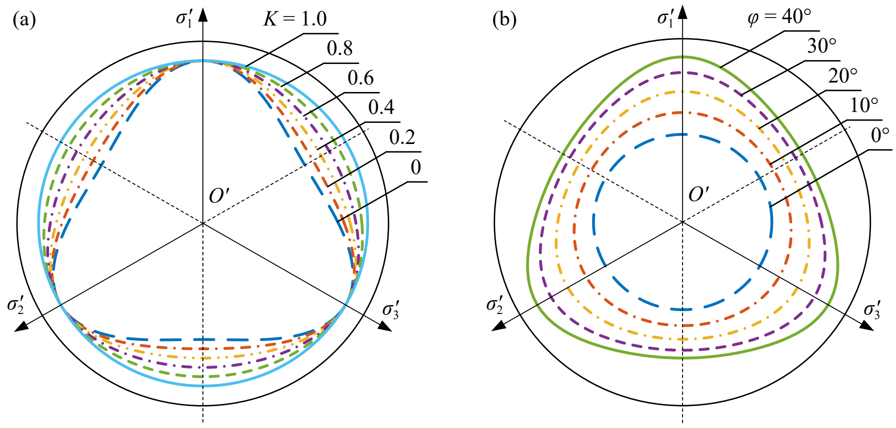

Fig. 4. **Failure envelope characteristics of the criterion proposed by Zhou and Li [40] on the deviatoric plane (I_1 = 100 kPa, c is the cohesion, c = 30 kPa, K is the ratio of triaxial tensile–compressive strengths in the deviatoric plane): (a) φ = 30°; (b) K = 0.5. Reprinted with permission from [40].**

2.4. GNFCs capable of degenerating into two criteria

2.4.1. GNFCs between the von Mises and MC criteria

Zhang *et al*. [41] proposed a frozen silt soil failure criterion by modifying the slope of the q–p curve on the meridian plane to explore the change regularity of the frozen soil strength, expressed as

$$q = k_1 g(\theta_\sigma) \tag{13}$$

with the included terms in Eq. (13) defined as

$$k_1 = M_0 p + c_0 \left[1 + \left(\frac{p}{c_1}\right)^v\right] \exp\left[-\left(\frac{p}{c_1}\right)^v\right] \tag{14}$$

$$g(\theta_\sigma) = \frac{\dfrac{(1-c_2)(3-\sin\varphi)}{3\left[1+(1/\sqrt{3})\tan\theta_\sigma \sin\varphi\right]} + c_2}{c_2 + (1-c_2)\cos\theta_\sigma} \tag{15}$$

where M_0 is the initial stress ratio, c_0 is the initial cohesion, c_1 and v are material parameters, φ is the internal friction angle, c_2 is the control parameter ranging in [0, 1], and k_1 is the deviatoric stress value in the meridian plane. $g(\theta_\sigma)$ is a deviatoric plane function.

When $c_2 = 0$, the criterion proposed by Zhang *et al.* [41] degenerates into the MC criterion, and the shape of the failure envelope on the deviatoric plane is an irregular hexagon. When $c_2 = 1$, this criterion degenerates into the von Mises criterion, and the shape becomes circular. When $0 < c_2 < 1$, the failure envelope is approximately pear-shaped.

Liu *et al*. [42] established the failure criterion of frost soil by considering the nonlinearity and asymmetry of frozen soil strength on the meridian plane and introducing the deviatoric plane function. When $s_1 = 0$, the criterion proposed by Liu *et al*. [42] degenerates into the MC criterion with an irregular hexagon failure envelope on the deviatoric plane. However, when $s_1 = 1$, this criterion degenerates into a circular von Mises criterion (Fig. 5). This criterion applies to the frozen soil strength changes of the three types: front peak, symmetric, and back peak.

2.4.2. GNFCs between the DP (or von Mises) and MN criteria

The GNFC failure envelope (Table 1 [43–51]) between DP (or von Mises) and MN criteria is a series of continuous, smooth, and convex curves. The criteria cover the DP (or von Mises) and MN criteria, which can reflect the strength characteristics of metal and friction materials. However, some mathematical expressions, such as Du *et al*. [44], who proposed a generalized von Mises and MN unified (GMMNu) criterion, are complex and challenging to separate stress invariants. Due to space limitations, this paper only lists the deviatoric plane shapes of the criteria proposed by Liu *et al.* [46] and Lu *et al*. [49] (see Fig. 6).

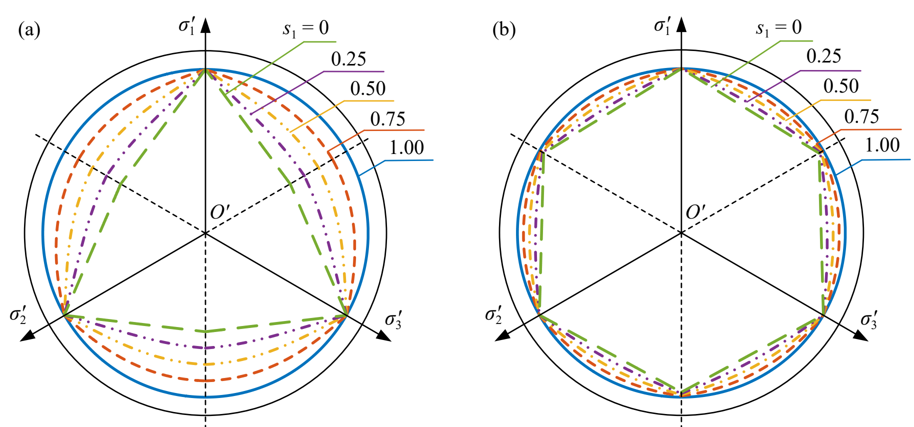

Fig. 5. **Failure envelope characteristics of the criterion proposed by Liu *et al.* [42] on the deviatoric plane (s_1 is an interpolation parameter in the range of [0, 1]): (a) φ = 48.485°; (b) φ = 2.612°. Reprinted from *Cold Reg. Sci. Technol.*, 161, X.Y. Liu, E.L. Liu, D. Zhang, *et al.*, Study on strength criterion for frozen soil, 1-20, Copyright 2019, with permission from Elsevier.**

Table 1. GNFCs between the DP (or von Mises) and MN

Presenter	Covered classic criteria	Commentary on the criterion
Yao *et al*. [43] and Du *et al*. [44]	Generalized von Mises and MN criteria	Can reasonably describe the nonlinear strength characteristics of various materials, such as metal, concrete, rock, sand, and clay; however, it is not easy to separate stress invariants
Xiao *et al*. [45]	Generalized von Mises and MN criteria	Can describe the anisotropy and nonlinear strength characteristics of soil
Liu *et al*. [46]	DP and MN criteria	Can predict the multiaxial strength of various geomaterials (Fig. 6(a))
Zhang *et al*. [47]	DP and extended DP [48] criteria	To solve the problem of the EDP that can only apply to the condition of the internal friction angle less than 22°
Lu *et al*. [49]	DP and MN criteria	The test shows that the Lu criterion can be applied to a series of geomaterials. However, only when the internal friction angle of the material is acquired can the parameter *b* (reflecting the IPS effect) be determined (Fig. 6(b))
Wan *et al*. [50]	Generalized von Mises and MN criteria	Can predict the failure behavior of soil, concrete, rock, and other materials and transform the traditional 2D constitutive model into a 3D model
Wan *et al*. [51]	Generalized von Mises and MN criteria	Can reflect the anisotropic properties of many materials, such as metal, rock, concrete, clay, and sand, with parameters with clear physical meanings

2.4.3. GNFCs between the DP (or von Mises) and LD criteria

Inspired by the GMMNu criterion [44], Liu *et al*. [52] proposed the generalized von Mises and Lade–Duncan (LD) unified criterion, which can degenerate into the generalized von Mises and LD criteria. Similar to the GMMNu criterion, this one cannot separate the stress invariants. Furthermore, He *et al*. [53] proposed a nonlinear 3D unified failure criterion for frozen soil to consider its composite properties and internal nonhomogeneity. Wang *et al*. [54] proposed a nonlinear unified failure criterion to capture the complex strength characteristics of geomaterials under 3D stress paths (Fig. 7).

The above GNFCs can degenerate into the generalized von Mises and LD criteria. Its failure envelope covers a series of continuous, smooth, and convex curves between the two criteria, which can describe the strength characteristics of various geomaterials.

2.4.4. GNFCs between the MN and LD criteria

Based on the MN and LD criteria, Mortara [55] derived a general expression for the stress tensor invariants of this criterion family. This criterion can degenerate into the MN and LD criteria (Fig. 8), which can reasonably describe the failure behavior of isotropic and cross-anisotropic materials. Furthermore, Xiao *et al*. [56] proposed a new failure criterion for various particulate materials based on the MN and LD criteria.

The failure envelope shapes of the above GNFCs cover all those between the MN and LD criteria, and the envelopes satisfy the requirements of smoothness and convexity.

2.5. GNFCs capable of degenerating into three criteria

2.5.1. GNFCs between the von Mises, MN, and LD criteria

Wan and Liu [57] proposed a new GNFC based on the von Mises, MN, and LD criteria (VML criterion), expressed as

$$\frac{I_1^{x-3y} I_2^{(3-x)/2}}{I_3^{1-y}} = c_3 \tag{16}$$

$$c_3 = \frac{(3-\sin\varphi)^{x-3y}(3+\sin\varphi)^{1.5-0.5x}}{(1+\sin\varphi)^{1-y}(1-\sin\varphi)^{0.5x-2y+0.5}} \tag{17}$$

where x, y, and c_3 are material constants.

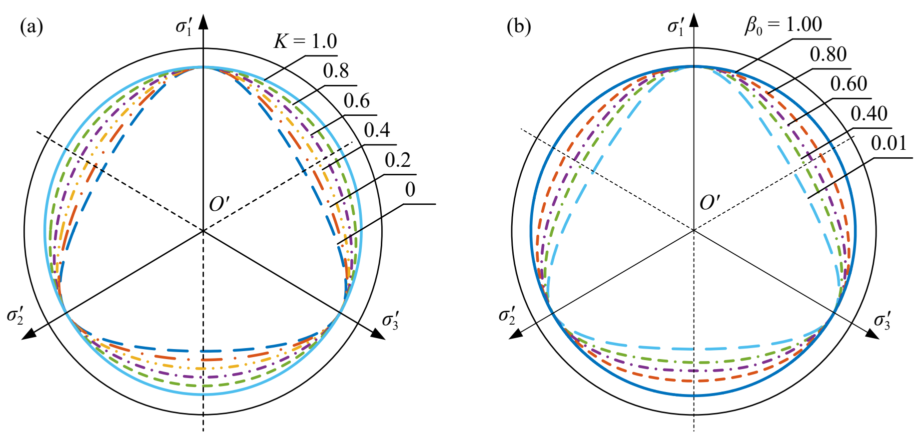

Fig. 6. Failure envelope characteristic of the GNFCs between the DP (or von Mises) and MN on the deviatoric plane: (a) criterion proposed by Liu *et al*. [46] (the material parameter ω = 0.7); (b) criterion proposed by Lu *et al*. [49] (the parameter β_0 reflects the IPS effect of geomaterials from 0 to 1). (a) Reprinted from *Int. J. Rock Mech. Min. Sci.*, 50, M.C. Liu, Y.F. Gao, and H.L. Liu, A nonlinear Drucker–Prager and Matsuoka–Nakai unified failure criterion for geomaterials with separated stress invariants, 1-10, Copyright 2012, with permission from Elsevier. (b) Used with permission of American Society of Civil Engineers from Development of a new nonlinear unified strength theory for geomaterials based on the characteristic stress concept, D. Lu, C.J. Ma, X.L. Du, *et al.*, 17, No. 04016058, 2017; permission conveyed through Copyright Clearance Center, Inc.

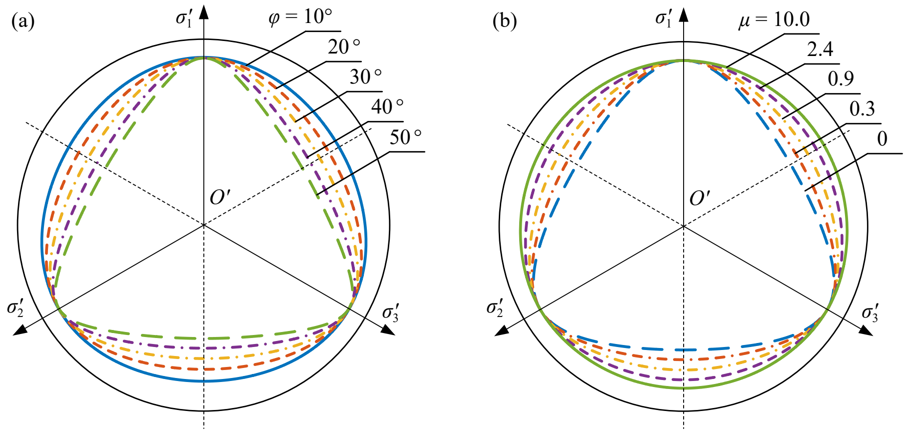

Fig. 7. Failure envelope characteristic of the criterion proposed by Wang *et al.* [54] on the deviatoric plane (μ is the combination coefficient, ranging from −0.3 to 10): (a) $\alpha = 0.1$; (b) $\varphi = 35°$.

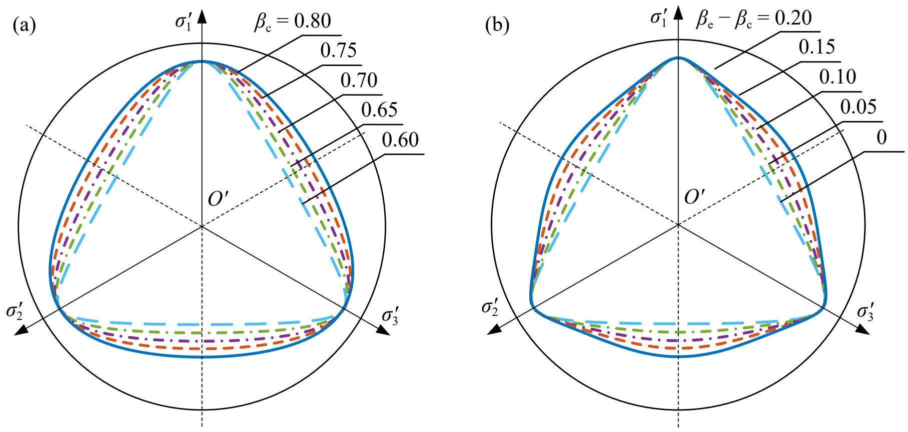

Fig. 8. Failure envelope characteristic of the criterion proposed by Mortara [55] on the deviatoric plane (β_e and β_c are material parameters): (a) $\beta_e - \beta_c = 0$; (b) $\beta_c = 0.60$. G. Mortara, *Int. J. Numer. Anal. Methods Geomech.*, 34, 953-977(2009), Copyright Wiley-VCH Verlag GmbH & Co. KGaA. Reproduced with permission.

When $x = 1$ and $x = 3$, the VML criterion degenerates into the MN and LD criteria, respectively. As x increases, the influence of the Lode angle on the VML criterion decreases. When $x = 100$, the shape of the failure envelope is close to the circle of the von Mises criterion. When $y = 2$, with the increase in x, the shape of the failure envelope gradually changes from convex to concave. When $x = 7$, the failure envelope does not meet the convexity requirements.

2.5.2. GNFCs between the MC, DP, and Mogi–Coulomb criteria

Based on the characteristics of peak failure strength of deep hard rocks, Feng *et al.* [58–59] proposed a 3D hard rock failure criterion (3DHRFC), expressed as

$$\left(\frac{\sigma_1 - \sigma_3}{\sin\varphi_b}\right)^2 = (\sigma_1 + \sigma_3 + 2c_4\cot\varphi_0)^2 + a_0 \tag{18}$$

When $a_0 = 0$, the 3DHRFC degenerates into a linear expression,

$$\sqrt{J_2} = \left(\frac{2\sqrt{3}I_1\sin\varphi_0}{3(3-\sin\varphi_0)} + \frac{2\sqrt{3}c_4\cos\varphi_0}{3-\sin\varphi_0}\right)g(\theta_\sigma) \tag{19}$$

where

$$g(\theta_\sigma) = \frac{3-\sin\varphi_0}{\sqrt{3}\cos\theta_\sigma\sqrt{S\left(\sqrt{3}\tan\theta_\sigma+1\right)^2 - 2\sqrt{3}\tan\theta_\sigma + 2 + \left(2\sqrt{3}\cos\theta_\sigma - 3\right)T\sin\varphi_0 + 2\sin\theta_\sigma\sin\varphi_0}} \tag{20}$$

$$\sin\varphi_b = \frac{\sin\varphi_0}{\sqrt{1-b+Sb^2} + T\left(1-\sqrt{1-b+b^2}\right)\sin\varphi_0} \tag{21}$$

where a_0, S, and T are material constants, c_4 is the cohesion of the rock peak failure strength under generalized triaxial compression, and φ_b is the internal friction angle corresponding to different IPS coefficients b. When $b = 0$, $\sin\varphi_b = \sin\varphi_0$.

Research shows that when $a_0 = 0$ and $b = 0$, the 3DHRFC degenerates into the MC criterion. Therefore, the linear expression of the 3DHRFC can be regarded as the 3D extension of the MC criterion. When $a_0 = 0$ and $g(\theta_\sigma) = 1$, the 3DHRFC degenerates into the DP criterion, and when $a_0 = 0$, $S = 1$, and $T = 0$, it degenerates into the Mogi–Coulomb criterion. The failure envelope of the linear 3DHRFC at the corner points makes it challenging to meet the smoothness requirement (Fig. 9 [58–59]).

2.6. GNFCs covering more than three criteria

2.6.1. GNFC applicable to geomaterials

Many scholars have proposed GNFCs (Table 2 [60–68])

Table 2. GNFCs of geomaterials degenerated into more than three classical criteria

Presenter	Covered classic criteria	Commentary on the criterion
Bigoni and Piccolroaz [60]	von Mises, DP, Tresca, modified Tresca, MC, modified Cam-clay, Rankine, and Ottosen	Can simulate the inelastic behavior of pressure–sensitive, frictional, ductile, and brittle–cohesive materials; however, the physical meaning of the introduced two new parameters β_1 and γ_1 is unclear (Fig. 10). Parameters β_1 and γ_1 model the shape of the deviatoric plane
Mortara [61]	MN and LD criteria, and criteria proposed by Lade [62], Kim and Lade [63], and Houlsby [64]	Can describe the strength behavior of clays, sands, and rocks quite well
Qiu *et al.* [65]	von Mises, Tresca, MC, DP, MSDPu [38,66], and generalized polyaxial strain energy strength criteria [67]	The criterion has a higher fitting accuracy than the MC, HB, modified Wiebols–Cook [68], modified Lade, DP, and Mogi criteria. The specific behaviors of some hard rocks need more in-depth experimental study to be revealed (Fig. 11)

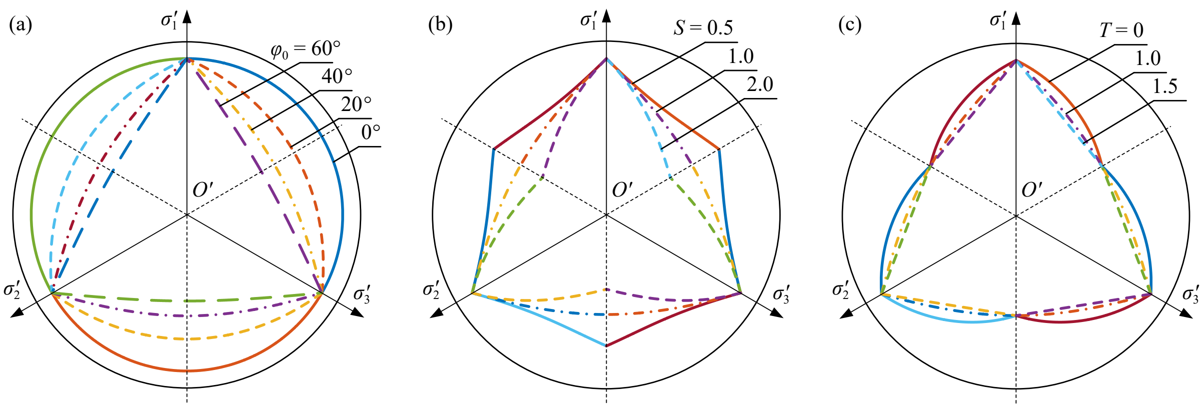

Fig. 9. Failure envelope characteristic of the 3DHRFC on the deviatoric plane (c_4 = 28 MPa) [58–59]: (a) S = 1 and T = 1; (b) φ_0 = 41.5° and T = 1; (c) φ_0 = 41.5° and T = 1.

that can be degenerated into more than three criteria and strived to use a unified mathematical expression to include or approximate the existing classical criteria. These criteria lay the foundation for the unified strength theory system. Due to space limitations, this paper only lists the deviatoric plane shapes of the criteria proposed by Bigoni and Piccolroaz [60] and Qiu *et al.* [65] (see Figs. 10 and 11).

2.6.2. GNFC for polygon-shaped deviatoric plane

The envelope shapes of most failure criteria in the above classical GNFCs are primarily circular, regular hexagons, and curved triangles. Many scholars improve the deviatoric plane function or introduce new parameters based on existing criteria to establish GNFCs, which have become a hotspot and development tendency of GNSTs.

This section introduces a class of GNFCs. The deviatoric plane shapes of these criteria can cover various forms, including triangles, hexagons, dodecagons, quadrangles, and circles (Table 3 [69–70]). It should be highlighted that because the expressions of some criteria mentioned above are rather complex and the deviatoric planes are mostly piecewise linear shapes, these criteria have yet to be applied in geomaterials and only serve to ensure the integrity of the criteria.

2.7. Other types of GNFCs

The envelope shapes on the deviatoric plane for the GNFCs described in this section are close or approximate to those of the classical criteria (Table 4 [71–76]). These criteria have theoretical significance in describing the strength characteristics of geomaterials. Due to space limitations, this paper only lists the deviatoric plane shapes of the criteria proposed by Ehlers [71] and Li and Tang [72] (see Fig. 12).

Some GNFCs proposed by many other scholars are not presented due to limited space, such as the new isotropic failure criterion [77] for friction materials, egg shape criterion [78], 3D failure criterion for rocks considering brittle and ductile domains [79], MN–LD failure criterion [80], anisotropic failure criterion of geomaterials [81], modified Griffith criterion [82], modified MC criterion [83–84], and unified strength theory [85].

Nowadays, all underground engineering construction should research the basic characteristics of geomaterials, especially the mechanical properties, including the nonlinear and deformation behavior of geomaterial strength under high stress and complex stress path conditions. Most existing classical GNFCs have been developed to the degree that they can be applied to various materials, making the failure envelope on the deviatoric plane meet the requirements of smoothness and convexity and covering or approximating more classical criteria by a unified mathematical expression. Some scholars have also developed corresponding constitutive models based on classical GNFCs. However, the study on the classical GNFC stays at the sample scale of geomaterials, and the material parameters are determined empirically at the sample scale or based on the classical criterion. No reports exist on applying classical GNFCs to the strength of engineering rock mass; therefore, the existing strength problems of rock mass are still based on the HB (or improved HB) empirical criterion.

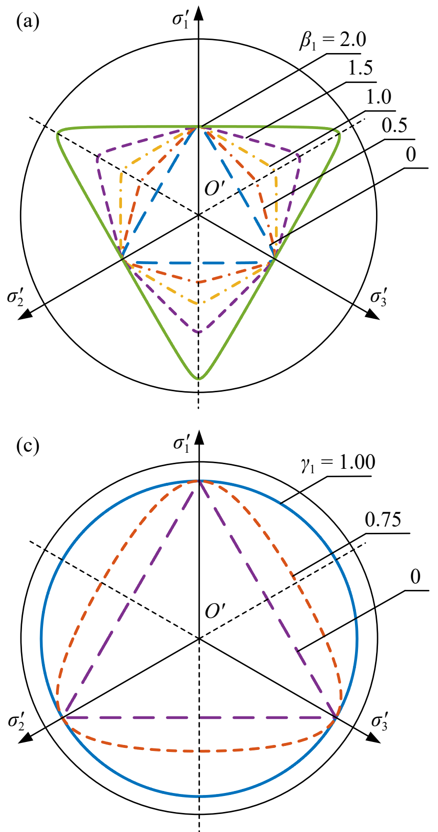

Fig. 10. **Failure envelope characteristics of the criterion proposed by Bigoni and Piccolroaz [60] on the deviatoric plane: (a) $\gamma_1 = 0.99$; (b) $\gamma_1 = 1$; (c) $\beta_1 = 0$; (d) $\beta_1 = 0.5$. Reprinted from *Int. J. Solids Struct.*, 41, D. Bigoni and A. Piccolroaz, Yield criteria for quasibrittle and frictional materials, 2855–2878, Copyright 2004, with permission from Elsevier.**

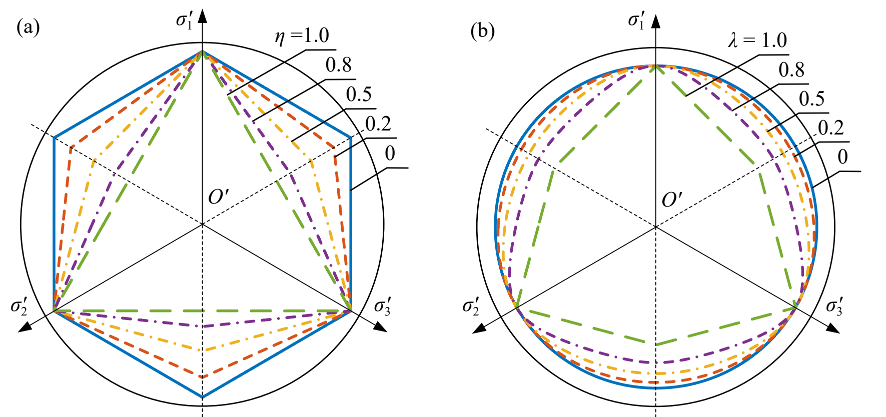

Fig. 11. **Failure envelope characteristics of the criterion proposed by Qiu *et al.* [65] on the deviatoric plane (the parameter η determines the edge number of the polygon, and parameter λ controls the transition degree of K and the failure envelope from polygon to circular): (a) $\lambda = 1$; (b) $\eta = 0.5$. Reprinted with permission from [65].**

Table 3. **GNFCs of polygon envelopes on the deviatoric plane [69–70]**

GNFC	Covered basic criteria
Modified Yu criterion	MC, Pisarenko-Lebedev, twin–shear strength, Tresca, Schmidt-Ishlinsky, and Sokolovsky criteria
Podgórski criterion	Ivlev, Mariotte, Tresca, and von Mises criteria
Modified Altenbach-Zolochevsky criterion	Ivlev, Mariotte, Tresca, Schmidt-Ishlinsky, and Sokolovsky criteria
Universal criterion of trigonal symmetry	Ivlev, Mariotte, Tresca, Schmidt-Ishlinsky, Sokolovsky, and von Mises criteria
Universal criterion of hexagonal symmetry	Ivlev, Mariotte, Tresca, Schmidt-Ishlinsky, Sokolovsky, Shlinsky-Ivlev, Rosendahl 1, Rosendahl 2, and von Mises criteria

Table 4. Other types of GNFCs

Presenter	Commentary on the criterion
Ehlers [71]	Can describe the strength behavior of brittle and granular materials; the deviatoric plane shape is close to a triangle with a rounded corner (Fig. 12(a))
Li and Tang [72]	Can reflect the cross-effect of a single principal stress on the sliding surface; however, it does not degenerate into MN and LD criteria (Fig. 12(b))
Shi *et al*. [73]	The failure envelope of the criterion is between the LD and MN criteria and is not circumscribed to the MC criterion; for coarser rockfill materials, the applicability of the criterion must be further verified
Liang and Li [74]	The failure envelope of the criterion satisfies the smoothness and convexity requirements but cannot degenerate into other classical criteria
Zhen and Li [75]	The 3D nonlinear failure criterion (3DNFC) can characterize the anisotropy of rock strength, and its deviatoric plane shape is similar to that of the MC and HB criteria
Gao *et al*. [76]	The deviatoric plane shape of the criterion cannot fully meet the convexity and smoothness requirements

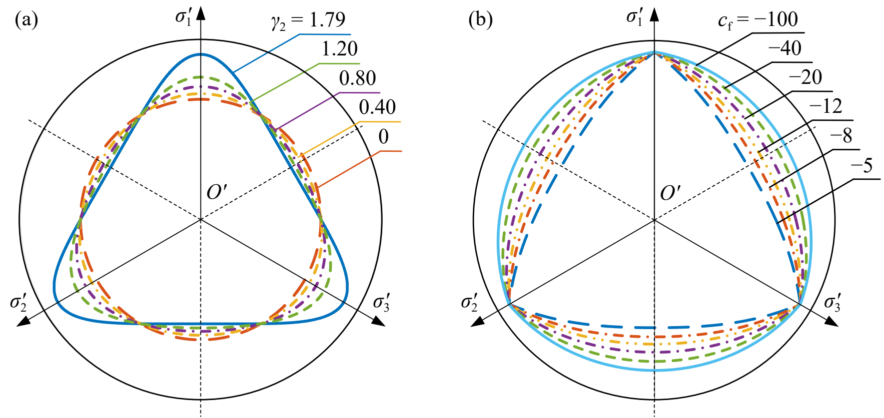

Fig. 12. Failure envelope characteristics of the other GNFCs on the deviatoric plane: (a) criterion proposed by Ehlers [71] (γ_2 and m_2 are the material parameters, and m_2 = 0.54545 for this paper); (b) criterion proposed by Li and Tang [72] (c_f is the material parameter, and I_1 = 50 MPa). (a) Adapted from [71].

3. Research status of generalized nonlinear 3D HB criteria

Over the past 40 years, numerous scholars have modified and improved the HB criterion to form a relatively complete theoretical system. This criterion is widely used in rock mechanics and engineering and can reflect the inherent nonlinear failure characteristics of rock and rock mass. This section systematically introduces some early nonlinear empirical, HB, and modified 3D HB criteria.

3.1. Early two-dimensional (2D) nonlinear empirical criteria

Many scholars have proposed various nonlinear empirical criteria to study the triaxial failure characteristics of rock materials. The early nonlinear empirical criteria are divided into three categories (Table 5 [86–94]). It is found that they do not consider the influence of IPS.

Among the early nonlinear empirical criteria, Hoek and Brown [93,95] first highlighted that the HB criterion, with a long history, is the most studied and widely used. This section discusses the 2D HB criterion.

Based on the nonlinear Griffith criterion [96] and the results of a wide range of triaxial tests on intact rock samples, Hoek and Brown [93,95] proposed the HB criterion for the first time, expressed as

$$\sigma_1 = \sigma_3 + \sigma_c \left(m_i \frac{\sigma_3}{\sigma_c} + 1 \right)^{0.5} \tag{22}$$

where σ_1 and σ_3 are the major and minor principal stresses, respectively, σ_c is the uniaxial compression strength of the intact rock, and m_i is the material constant.

Later, Hoek [97] and Hoek *et al*. [98] proposed the generalized HB (GHB) criterion to estimate the strength of rock mass, expressed as

Table 5. Early 2D nonlinear empirical criteria

Presenter of criterion	Description of criterion type	Commentary on the criterion
Murrell [86], Bieniawski [87], Yudhbir *et al*. [88], and Sheorey *et al*. [89]	Category I, the major principal stress σ_1 is a function of the minor principal stress σ_3.	The early 2D nonlinear empirical criteria are simple in form, where the material parameters are determined by the statistical curve fitting of uniaxial compressive and tensile strengths and test results. However, they fail to consider the influence of IPS on rock strength.
Hobbs [90] and Ramamurthy and Arora [91]	Category II, the major principal shear stress $\tau_{13} = (\sigma_1 - \sigma_3)/2$ is a function of the minor principal stress σ_3.	
Franklin [92], Hoek and Brown [93], and Yoshida *et al*. [94]	Category III, the major shear stress $\tau_{13} = (\sigma_1 - \sigma_3)/2$ is a function of the normal stress $\sigma_{13} = (\sigma_1 + \sigma_3)/2$.	

$$\sigma_1 = \sigma_3 + \sigma_c\left(m_b \frac{\sigma_3}{\sigma_c} + s\right)^a \tag{23}$$

where m_b, s, and a are the rock mass material constants. They can be expressed by the geological strength index and the factor D of the disturbance degree due to blast damage and stress relaxation [99].

Fig. 13 [97–98] shows that the failure envelope of the GHB criterion on the deviatoric plane is an irregular hexagon. When $\theta_\sigma = 0°$ and $\theta_\sigma = 60°$, the failure envelope of the criterion does not meet the smoothness requirement, and the influence of the IPS is not considered. The true triaxial test of rock shows [100–102] that the IPS significantly influences rock strength.

3.2. Improved HB criteria based on the deviatoric plane function

Since the HB criterion does not meet the smoothness requirement, some scholars have retained the nonlinear characteristics of the original meridian plane and adopted a smooth and convex deviatoric plane function to modify the HB criterion (Table 6 [103–106]). The modified 3D HB criteria are expressed as

$$\sqrt{J_2} = \frac{-m_i\sigma_c + \sqrt{m_i^2\sigma_c^2 + 12m_i\sigma_c I_1 + 36\sigma_c^2}}{6\sqrt{3}} g(\theta_\sigma) \tag{24}$$

These 3D HB criteria retain the nonlinear characteristics of the original meridian plane of the HB criterion and solve the smoothness requirement problem. It is worth mentioning that, at the corners, the criterion proposed by Yang *et al.* [105] does not fully meet the smoothness requirement (Fig. 14(a)). The envelope shape (Fig. 14(b)) of the deviatoric plane function of the criterion proposed by Li [106] can be determined based on the material constant m_i and test data.

3.3. Generalized nonlinear 3D HB criteria established by the construction method

The generalized nonlinear 3D HB criterion established by the construction method can be expressed as

$$\frac{1}{m_b\sigma_c^{1/a-1}}\left(\sqrt{3J_2}\right)^{1/a} + \frac{1}{\sqrt{3}}D(\theta_\sigma)\sqrt{J_2} - \frac{s\sigma_c}{m_b} - \frac{I_1}{3} = 0 \tag{25}$$

where $D(\theta_\sigma)$ is a function of θ_σ.

The existing 3D HB criteria for this category are shown in Table 7 [107–118]. The GZZ and new GZZ criteria can degenerate into HB under triaxial compression and tensile conditions. The 3D HB criteria constructed by Zhang and Zhu [108], Zhang [109], Jiang *et al.* [110], Jiang and Xie [111], Jiang and Zhao [112], and Cai *et al.* [115–118] are challenging to meet the smoothness and convexity requirements. The new criterion C is similar to the new GZZ criterion. Furthermore, the Pan–Hudson criterion does not consider the influence of Lode angle θ_σ. The rock strengths predicted by this criterion are the same under triaxial compression and tension; therefore, the tensile strength of rock material is overestimated. Due to space limitations, this paper only lists the deviatoric plane shapes of the criteria proposed by Zhang [109] and Jiang [113] (see Fig. 15).

It is worth mentioning that Chen *et al.* [119–120] proposed a modified GZZ (MGZZ) criterion based on the average effective stress. The criterion satisfies the convexity requirement and has an explicit expression. It can be conveniently used for theoretical and numerical analysis. Furthermore, it can degenerate into the HB criterion under triaxial compression and tension.

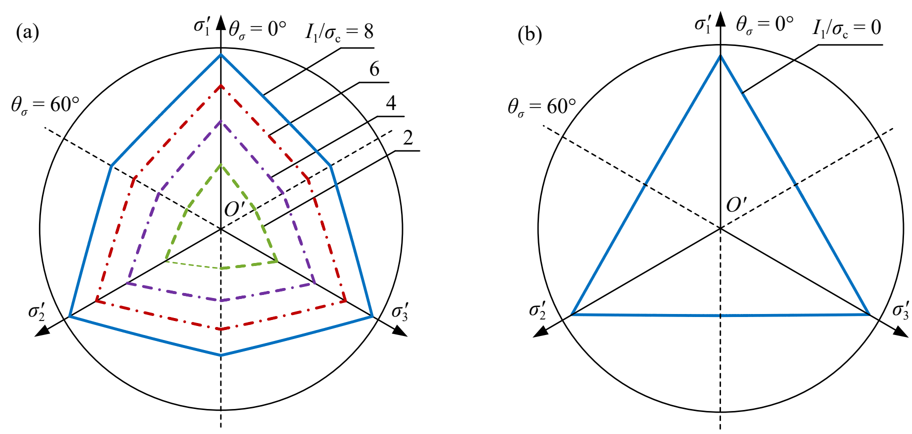

Fig. 13. Failure envelope characteristics of the GHB criterion on the deviatoric plane [97–98] (σ_c = 100 MPa, m_i = 13.29, a = 0.5, and s = 0.329): (a) I_1 = 200, 400, 600, and 800 MPa; (b) I_1 = 0 MPa.

Table 6. Modified HB criteria based on the deviatoric plane function

Presenter	Characteristics of the deviatoric plane function
Lee *et al.* [103], Zhang *et al.* [104]	The shape of the function changes continuously from triangle to circle, and the failure envelope meets the convexity requirement. Only when $0° < \theta_\sigma < 30°$, the curvature of the envelope shows a slight difference.
Yang *et al.* [105]	The shape of the function changes continuously from MC to DP criteria (Fig. 14(a)).
Li [106]	The shape of the function changes continuously from a curved triangle to a circle, meeting the requirements of smoothness and convexity (Fig. 14(b)).

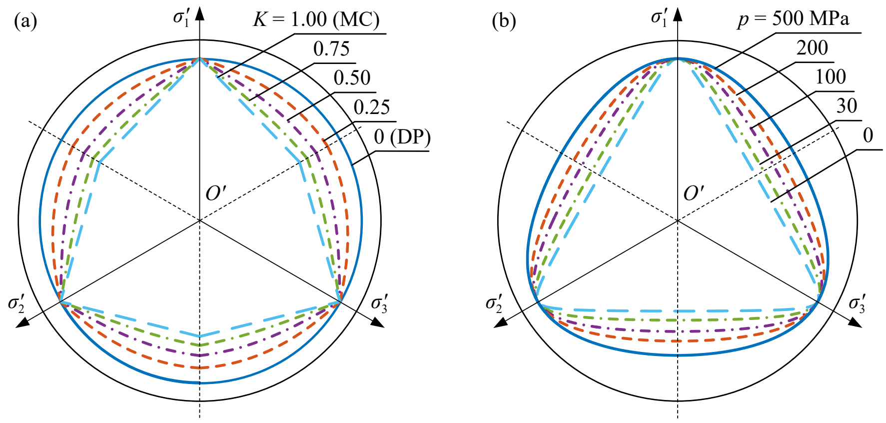

Fig. 14. Failure envelope characteristics of the modified HB criteria based on the deviatoric plane function on the deviatoric plane: (a) criterion proposed by Yang *et al.* [105]; (b) criterion proposed by Li [106]. (a) Reprinted from *Cold Reg. Sci. Technol.*, 86, Y.G. Yang, F. Gao, and Y.M. Lai, Modified Hoek–Brown criterion for nonlinear strength of frozen soil, 98-103, Copyright 2013, with permission from Elsevier; (b) Reprinted with permission from [106].

3.4. Improvement of the HB criterion by introducing principal stress weight combination

This section addresses the improvement of the HB criterion based on the weight combination of the IPS σ_2 and minor principal stress σ_3 (Table 8 [121–127]). This improved HB criterion shares the same material constants with the GHB criterion, inheriting the advantages of the GHB criterion parameters. However, they do not meet the smoothness requirement and are inconvenient for numerical calculation. Due to space limitations, this paper only lists the deviatoric plane shapes of the criteria proposed by Li *et al.* [123] and Gao *et al.* [126] (see Fig. 16).

3.5. Improved HB criterion based on other theories (criteria)

3.5.1. Combination of the HB criterion with other criteria

Yu *et al.* [128] proposed a GNFC (Fig. 17(a)) for rock materials by combining the unified strength theory with the HB criterion, which can be extended to rock mass strength. Priest [129] proposed a comprehensive criterion combining the DP and HB criteria. Nevertheless, the criterion has challenges in reflecting the strength difference of rock under triaxial compression and tension [130]. Benz *et al.* [131–132] presented an improved HB criterion influenced by the shape of the Lade, Matsuoka, and Nakai deviatoric plane function. Huang *et al.* [133] combined the HB criterion with the GNFC and developed a modified HB criterion.

Vicente da Silva and Antão [134] proposed a new HB–MN failure criterion (Fig. 17(b)), which extends the 2D HB criterion using the failure envelope of the MN criterion on the deviatoric plane. However, this criterion makes it challenging to separate the stress invariants. Based on the 2D MC and HB criteria, Schwartzkopff *et al.* [135] set up an HB criterion under general stress conditions, of which the failure envelope meets the smoothness and convexity requirements on the deviatoric plane, conductive to the development and application of numerical simulation software.

3.5.2. Combination of the HB criterion with other theories

Based on the linear elastic fracture theory, Zuo *et al.* [136–137] derived the HB criterion using a 3D crack model. They found, for the first time, that the constant m_i in the HB criterion has physical significance; it is related to the ratio of uniaxial compressive–tensile strengths.

Based on the fracture mechanics theory and the wing crack model, Wang *et al.* [138] derived a 3D strength criterion for hard rocks. Their research shows that when $\sigma_2 = \sigma_3$, this criterion can degenerate into the criterion proposed by

Table 7. Modified 3D HB criteria under different functions $D(\theta_\sigma)$

Presenter	$D(\theta_\sigma)$	Name
Pan and Hudson [107]	1.5	Pan–Hudson criterion
Zhang and Zhu [108]; Zhang [109]	$(3+2\sin\theta_\sigma)/2$ or $[3-2\cos(\pi/3+\theta_\sigma)]/2$	Zhang–Zhu criterion and GZZ criterion (Fig. 15(a))
Jiang *et al.* [110]	$3-2\cos\theta_\sigma$	New criterion A
Jiang *et al.* [110]	$[5-2\cos(2\theta_\sigma)]/3$	New criterion B
Jiang and Xie [111]; Jiang *et al.* [110]	$[3-\cos(3\theta_\sigma)]/2$	New criterion C
Jiang and Zhao [112]	$2\cos[(\pi/3)-\theta_\sigma]$	New criterion D
Jiang [113]	$\cos\left[\frac{1}{3}\arccos(k\cos(3\theta_\sigma))\right]/\cos\left[\frac{1}{3}\arccos(k)\right]$	New criterion E (Fig. 15(b))
Jiang [114]	1.618–1.691	
Cai *et al.* [115–118]	$[3+\sin(3\theta_\sigma)]/2$	New GZZ criterion

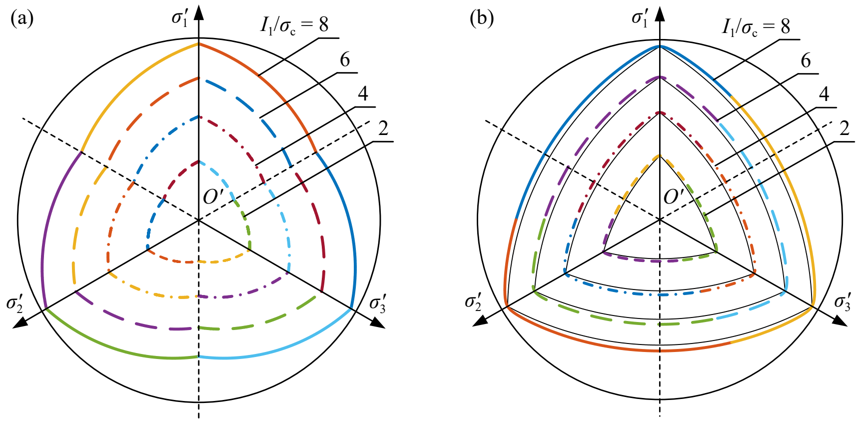

Fig. 15. Failure envelope characteristics of the GZZ criterion, new criterion E, and MGZZ criterion on the deviatoric plane (σ_c = 100 MPa, m_i = 13.29, a = 0.5, and s = 0.329): (a) GZZ criterion [109]; (b) new criterion E (the black smooth curved triangle represents the shape of the new criterion D) [113]. (a) Adapted from [109]. (b) Adapted from [113].

Table 8. Improved HB criteria based on the weight combination of σ_2 and σ_3

Presenter	Improved method	Commentary on the criterion
Singh *et al.* [121]	The minimum principal stress σ_3 is replaced by $(\sigma_2 + \sigma_3)/2$.	When I_1 = 0 MPa, the envelope shape of the Singh criterion is circular, and as the hydrostatic stress increases, the envelope shape on the deviatoric plane gradually transits from a circle to a curved triangle. The criterion envelope under triaxial compression does not meet the smoothness requirement.
Priest [122]	$\sigma'_{3HB} = \omega_1\sigma'_2 + (1-\omega_1)\sigma'_3$, where ω_1 is a weighting factor in the range of 0–1. σ'_2 and σ'_3 are the effective stresses. σ'_{3HB} is the minimum 2D HB criterion effective stress at failure.	The Priest criterion (i.e., the simplified Priest criterion) cannot degenerate into the HB criterion and does not meet the smoothness requirement.
Li *et al.* [123]	σ_3 can be replaced by the weighted average of σ_2 and σ_3, i.e., $(\sigma_1 + b_0\sigma_2)/(b_0 + 1)$ in the low σ_2 range, whereas σ_1 can be replaced by the weighted average of σ_1 and σ_2, i.e., $(\sigma_1 + b_0\sigma_2)/(b_0 + 1)$ in the high σ_2 range, where b_0 is the shape factor of the failure envelope varying in the range of 0–1.	The proposed criterion inherits the advantages of the HB criterion and has a straightforward mathematical expression. However, the envelope on the deviatoric plane is not smooth (Fig. 16(a)) and can introduce problems of misconvergence or slow convergence in numerical simulations.
Ma *et al.* [124]	σ_3 is replaced by $(n\sigma_2 + \sigma_3)$, where n is the material parameter in the range of 0–0.5.	The proposed criterion can degenerate into the HB criterion but does not meet the smoothness requirement.
Que *et al.* [125]	σ_3 is replaced by $(n_0\sigma_2 + \sigma_3)/(n + 1)$, where n_0 is the stress weighting factor, ranging from 0 to 1.	When $\sigma_2 = \sigma_3$, the proposed 3D version is automatically converted to the GHB criterion, which does not meet the smoothness requirement.
Gao *et al.* [126]	σ_3 is replaced by $[\alpha_1(\sigma_2 - \sigma_3) + 2\sigma_3]/2$, where α_1 is a parameter related to IPS, varying in 0–1.	The lower bound of the proposed criterion is the HB criterion, and the upper bound is the Singh criterion. This criterion can better reflect the influence of IPS under a complex stress state; however, the failure envelope on the deviatoric plane does not meet the smoothness requirement (Fig. 16(b)).
Shi *et al.* [127]	To enhance the IPS effect on rock strength, $(\beta_2\sigma_2 + \sigma_3)/(1 + \beta_2)$ is used instead of σ_3. For a weakening effect, $(\beta_2\sigma_2 + \sigma_1)/(1 + \beta_2)$ is used instead of σ_1, where β_2 is the IPS coefficient.	The proposed criterion inherits the advantages of the HB criterion and can well describe the influence of IPS on rock strength.

Zuo *et al.* [136]. However, the physical meanings and determination methods of the two additional parameters in this criterion need further verification. Furthermore, Zhou *et al.* [139] proposed a new 3D GNFC for rock materials based on the micromechanics method.

3.6. Improved HB criterion based on rock anisotropy or laboratory tests

The HB criterion is widely used for the strength prediction of rocks and rock masses but is less applicable to jointed rocks with significant anisotropy. Therefore, Saroglou and Tsimbaos [140] modified the HB criterion by introducing a new parameter, k_β, to describe the influence of rock strength anisotropy. Given the effects of anisotropy and hydration on the strength of unsaturated shale, Zhang *et al.* [141] proposed a modified HB criterion.

Subsequently, some scholars have modified the HB criterion based on laboratory tests [142–143]. For example, Peng and Cai [143] proposed an improved HB criterion based on the residual strength of rock. Some HB criteria revised by many scholars are not listed due to space limitations.

The generalized nonlinear 3D HB criterion can reflect the inherent nonlinear failure characteristics of rock and rock mass, introducing a new way to study rock mass strength. In the past 40 years, at least 30 expressions of the HB criterion have come into being. Until now, scholars have constantly

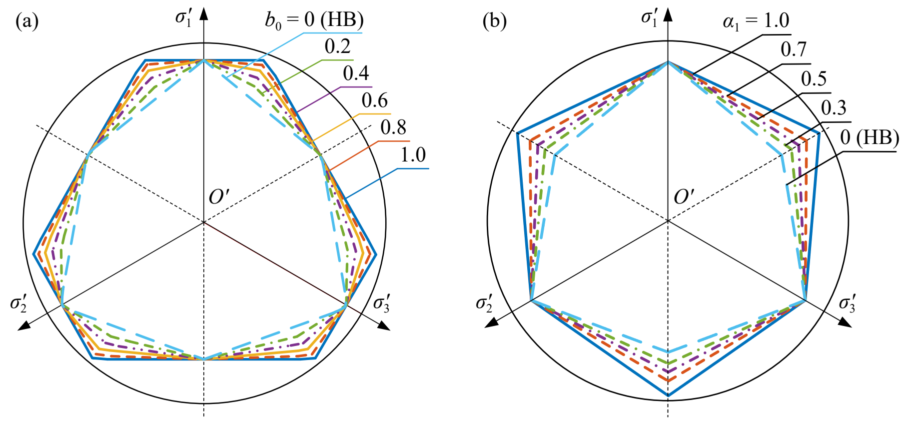

Fig. 16. Failure envelope characteristics of the improved HB criteria based on the weight combination of σ_2 and σ_3 on the deviatoric plane: (a) criterion proposed by Li *et al.* [123] (b_0 is defined as the IPS coefficient to evaluate the IPS effect); (b) criterion proposed by Gao *et al.* [126] (p = 800 MPa, σ_c = 99.13 MPa, m_b = 11.4, s = 1, and a = 0.5). (b) Used with permission of American Society of Civil Engineers from Novel 3D failure criterion for rock materials, F. Gao, Y.G. Yang, H.M. Cheng, *et al.*, 19, No. 04016058, 2017]; permission conveyed through Copyright Clearance Center, Inc.

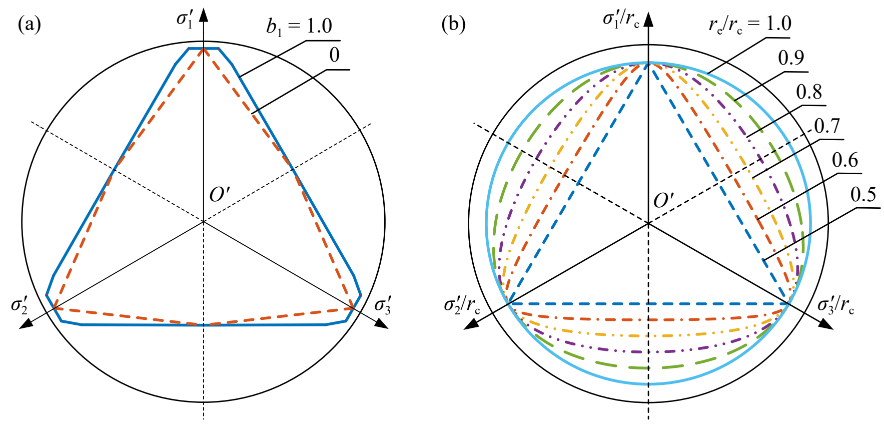

Fig. 17. Failure envelope characteristics of the combination of the HB criteria with other criteria on the deviatoric plane: (a) criterion proposed by Yu *et al.* [128] (b_1 is the unified strength theory parameter ranging in [0, 1]); (b) criterion proposed by da Silva and Antão [134] (I_1 = 200 MPa, parameters A = 835.72 MPa, and B = 6761.86 MPa^2, r_c is compression radius). (a) Reprinted from *Int. J. Rock Mech. Min. Sci.*, 39, M.H. Yu, Y.W. Zan, J. Zhao, *et al.*, A unified strength criterion for rock material, 975-989, Copyright 2002, with permission from Elsevier.

submitted various expressions. Among them are many innovative studies, and some have made breakthroughs in the theory. However, most modified HB criteria focus on intact rock strength. However, for the study of rock mass strength, the surrounding rock classification system is generally used to evaluate the rock mass quality, the rock mass parameters are obtained based on empirical methods, and then the rock mass strength is determined. Furthermore, in the research of rock mass parameters, although the digital *in-situ* values of rock mass parameters have been preliminarily realized, rock mass mechanics tests at an engineering scale are challenging, and the strength construction of *in-situ* rock mass in underground engineering is difficult to solve effectively.

4. Research progress of the GNFCs in the recent five years

Since 2017, the authors have conducted collaborative research on GNFCs and made significant progress. In this section, two deviatoric plane functions proposed by the authors will be systematically introduced, and then a class of GNFCs based on the two functions (with the meridional plane function being characterized by a simple power function) will be emphatically introduced. The function expression is

$$\sqrt{J_2} = N_f p_a \left(\frac{I_1 + \delta_t}{p_a} \right)^{n_1} g(\theta_\sigma) \tag{26}$$

where N_f reflects the influence of friction factors on the failure envelope (the friction characteristics). δ_t reflects the triaxial tensile strength of the material. n_1 is the hydrostatic stress effect index, and its range is [0.5, 1], reflecting the influence degree of hydrostatic stress.

4.1. MNGNF, MCNUF, and LDNUF criteria

4.1.1. MNGNF criterion

Based on the deviatoric plane function of the MN criterion, Wu *et al.* [144] and Zhang [145] introduced a new strength parameter and proposed a new two-parameter deviatoric plane function, expressed as

$$g(\theta_\sigma) = \frac{\sin\left(\frac{\pi}{3}\alpha_2 - \frac{1}{3}\sin^{-1}A_1\right)}{\sin\left\{\frac{\pi}{3}\alpha_2 + \frac{1}{3}\sin^{-1}[A_1\sin(3\theta_\sigma)]\right\}} \tag{27}$$

where A_1, ranging in [1, 1.5], is the function of the first invariant of the stress tensor I_1 or the internal friction angle φ. The parameter α_2, ranging in [0, 1], primarily controls K.

Research shows that the new two-parameter deviatoric plane function can realize the deviatoric plane shape of many classical criteria (Fig. 18 [144–145]), such as the DP, Tresca, MC, LD, MN, and Ottosen criteria, and meet the smoothness and convexity requirements.

4.1.2. MCNUF and LDNUF criteria

Wang *et al*. [146] introduced two parameters to extend the deviatoric plane function of the LD criterion and proposed a novel three-parameter deviatoric plane function, expressed as

$$g(\theta_\sigma) = \frac{-\sin\left(\frac{\pi}{6}\alpha_3+\beta_3+\frac{1}{3}\cos^{-1}\gamma_3\right)}{\sin\left\{\frac{\pi}{6}\alpha_3 + \frac{1}{3}\cos^{-1}[\gamma_3\cos(3\theta_\sigma)]\right\}} \tag{28}$$

where α_3 controls the parameter K, ranging in [1, 2]. β_3 primarily controls the shape and size of the deviatoric plane and takes one of six fixed values ($-\pi$, $-\pi/1.4353$, $-\pi/1.5$, $\pi/3.2971$, $\pi/3$, or π). γ_3 is related to the internal friction angle φ or the first invariant of the stress tensor I_1 in the range of [0, 1].

The novel three-parameter deviatoric plane function can generate the deviatoric plane shapes of many classical criteria (Fig. 19 [146]), including the Rankine, Tresca, von Mises, generalized Tresca, MC, DP, MN, LD, and Ottosen criteria. Moreover, it can cover the shapes of the deviatoric plane functions proposed by Bigoni and Piccolroaz [60], Qiu *et al.* [65], and Wu *et al*. [144] and satisfy the smoothness and convexity requirements.

This paper only takes the three-parameter deviatoric plane function (Eq. (28)) as an example for a brief analysis. Fig. 19(e) shows that the function is equivalent to the MC deviatoric plane function when $\alpha_3 \in [1, 2]$, $\beta_3 = \pi$ or $-\pi$, and $\gamma_3 = 1.0$. Fig. 19(i) shows that it is equivalent to the Ottosen deviatoric plane function when $\gamma_3 \in [0, 1]$, $\alpha_3 = 1.0$, $\beta_3 = \pi/3$, or $-\pi/1.5$, and Fig. 19(f)–(h), show that its shape degenerates into a DP circle when $\gamma_3 = 0$.

When $\alpha_3 \in [1, 2]$, the deviatoric plane shapes of the superposition of Fig. 19(a) and (e) are equivalent to the shape of Fig. 10(a) and those of Fig. 19(b) and (e) are equivalent to the shape of Fig. 10(b). When $\gamma_3 \in [0, 1]$, Fig. 19(f) and (g) shows that the deviatoric plane shapes are equivalent to those of Fig. 10(c) and (d), respectively, whereas those of Fig. 19(e) and (f) are equivalent to those of Fig. 11(a) and (b), respectively. Fig. 19(c)–(h) and Fig. 18(a)–(f) show that the three-parameter deviatoric plane function (Eq. (28)) is equivalent to the two-parameter one (Eq. (27)).

The studies above find that the deviatoric plane function mentioned by Bigoni and Piccolroaz [60], Qiu *et al*. [65], and Wu *et al*. [144] is only a unique case of the three-parameter

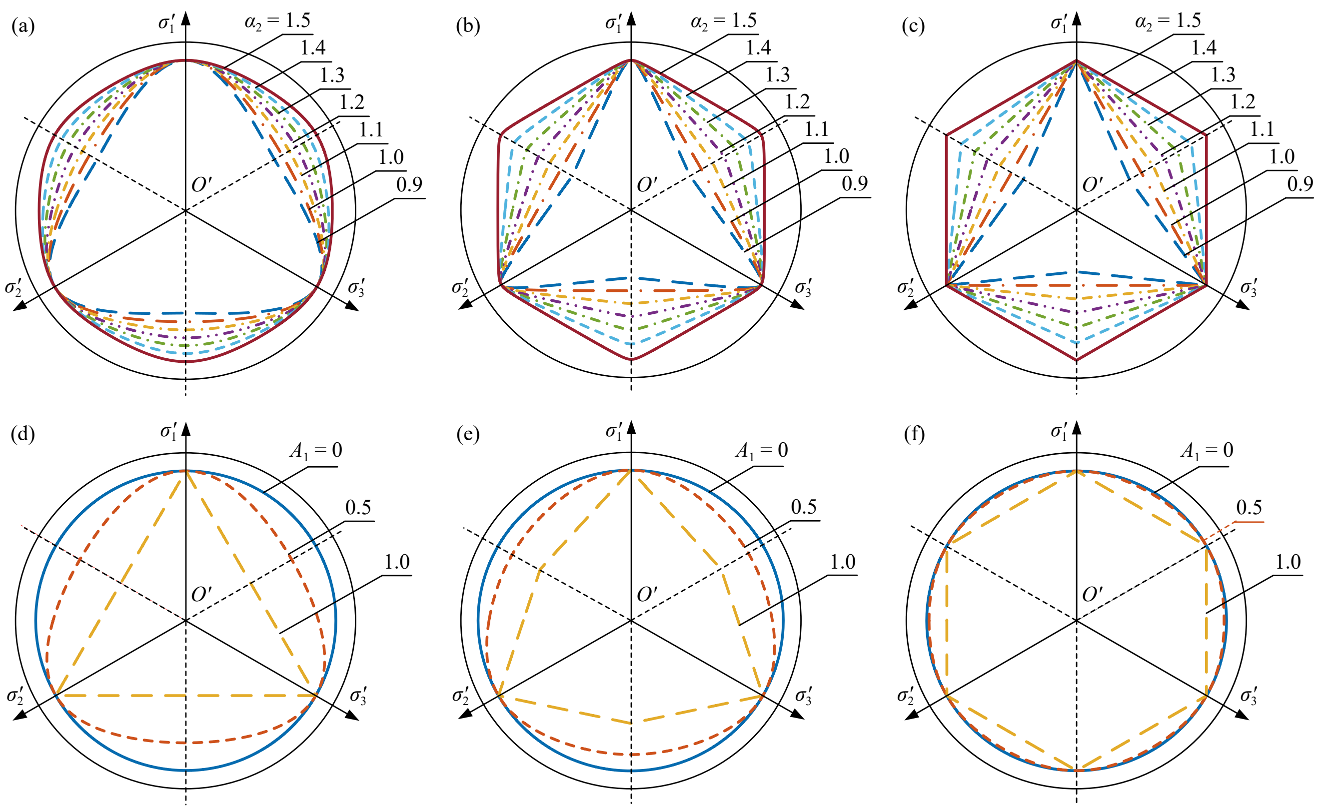

Fig. 18. Shape of the two-parameter deviatoric plane function changing with parameters A_1 and α_2 [144–145]: (a) A_1 = 0.7; (b) A_1 = 0.99; (c) A_1 =1; (d) α_2 = 1.0; (e) α_2 = 1.2; (f) α_2 = 1.5. Reprinted by permission from Springer Nature: *Acta Geotech.*, A generalized nonlinear failure criterion for frictional materials, S.C. Wu, S.H. Zhang, C. Guo, and L.F. Xiong, Copyright 2017.

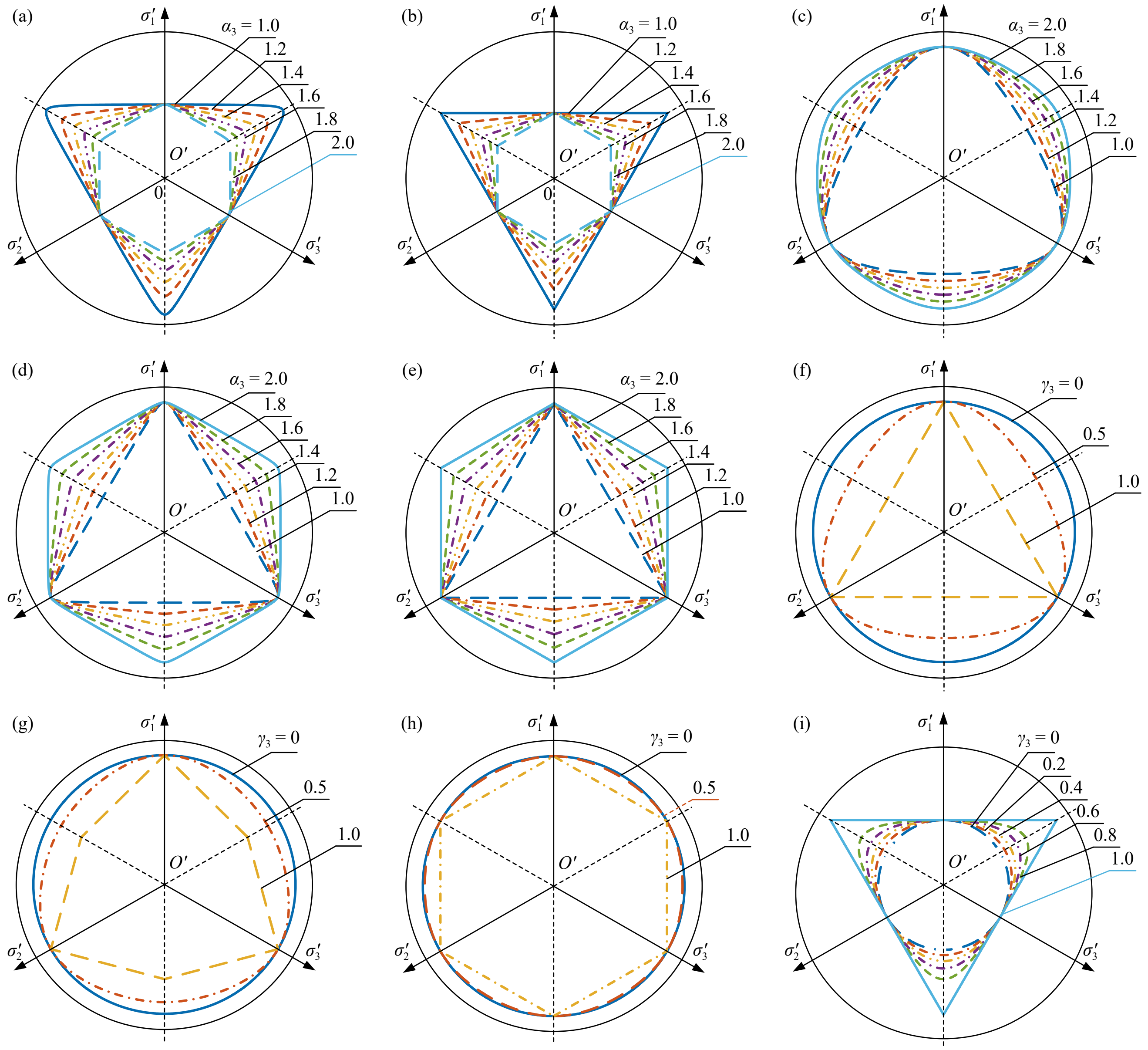

Fig. 19. Effects of the variation of α_3, β_3, and γ_3 on the failure envelope in the novel three-parameter deviatoric plane function [146]: (a) $\beta_3 = \pi/3.2971$ or $-\pi/1.4353$ and $\gamma_3 = 0.99$; (b) $\beta_3 = \pi/3$ or $-\pi/1.5$ and $\gamma_3 = 1.0$; (c) $\beta_3 = \pi$ or $-\pi$ and $\gamma_3 = 0.7$; (d) $\beta_3 = \pi$ or $-\pi$ and $\gamma_3 = 0.99$; (e) $\beta_3 = \pi$ or $-\pi$ and $\gamma_3 = 1.0$; (f) $\alpha_3 = 1.0$ and $\beta_3 = \pi$ or $-\pi$; (g) $\alpha_3 = 1.5$ and $\beta_3 = \pi$ or $-\pi$; (h) $\alpha_3 = 2.0$ and $\beta_3 = \pi$ or $-\pi$; (i) $\alpha_3 = 1.0$ and $\beta_3 = \pi/3$ or $-\pi/1.5$. Reprinted by permission from Springer Nature: *Acta Geotech.*, A novel three-dimensional nonlinear unified failure criterion for rock materials, J.X. Wang, S.C. Wu, X.K. Chang, *et al.*, Copyright 2023.

deviatoric plane function.

For the convenience of distinction, the 3D generalized nonlinear failure criterion constructed from Eq. (27) is named the MNGNF criterion. Based on Eq. (28), the 3D nonlinear unified failure (NUF) criteria constructed with parameter γ_3 using the basic parameters of the MC and LD criteria are named the MCNUF and LDNUF criteria, respectively.

Research shows that the MNGNF criterion can degenerate into several classical criteria, including the Tresca, circumscribed DP, MC, LD, and MN criteria. The MCNUF (or LDNUF) criterion can degenerate into the Rankine, Tresca, von Mises, generalized Tresca, MC, DP, MN, LD, and Ottosen criteria. The MCNUF (or LDNUF) criterion can degenerate into the criteria presented by Bigoni and Piccolroaz [60] and Qiu *et al*. [65] when the corresponding transformation is performed on the meridian plane function. Furthermore, the proposed MCNUF (or LDNUF) criterion is generalized.

By applying the proposed MNGNF, MCNUF, and LDNUF criteria to four rock materials, i.e., Yamaguchi marble, Laxiwa granite, Tien–Liao mudstone, and Castlegate sandstone, it was found that the three criteria have good prediction performances, with predictive errors of 2.3006%–3.8013%, 2.3500%–5.4100%, and 2.4200%–10.4834%, respectively. The proposed criteria can describe the hydrostatic stress, IPS effect, hydrostatic stress, and IPS coupling effect for various rock materials quite well [146].

4.2. MNHB and NGHB criteria

Wu *et al*. [147] and Zhang *et al*. [148] proposed a modified 3D HB criterion (the MNHB criterion) by revising the HB criterion by the new two-parameter deviatoric plane function (Eq. (27)) [144], which has the same expression as Eq. (24). The MNHB criterion meets the smoothness and convexity requirements. It retains the nonlinear characteristics of the original meridian plane in the HB criterion. However, it can only be applied to intact rock strength studies and cannot degenerate into the HB criterion.

Later, inspired by Eqs. (25) and (28), Wang *et al.* [17] introduced a new NGHB criterion, which has the same expression as Eq. (25), with the function $D(\theta_\sigma)$ being expressed as

$$D(\theta_\sigma)=\frac{\sin^{1/n_2}\left\{\frac{\pi}{6}+\frac{1}{3}\arccos\left[\beta_4\cos(3\theta_\sigma)\right]\right\}}{\sin\left[\frac{\pi}{6}+\frac{1}{3}\arccos(\beta_4)\right]} \tag{29}$$

where n_2 and β_4 are the shape factors of the deviatoric plane, which primarily control the shape of the deviatoric plane of the NGHB criterion under triaxial tension and compression, respectively, with the ranges of $\beta_4 \in [0, 1]$ and $n_2 \in [1, 4]$.

Fig. 20 shows the failure envelope characteristics of the NGHB criterion on the deviatoric plane [17]. The criterion unifies parts of the modified HB criteria into the same strength theoretical framework, such as Jiang and Zhao's [112], Jiang's [113], the HB criterion under triaxial compression–tension, and the Priest criterion [129] under triaxial compression. This criterion lays the foundation for further unifying the nonlinear 3D HB criterion.

The NGHB criterion can be applied to rock mass strength studies and is a generalized nonlinear 3D HB rock mass strength criterion in the true sense. It has better prediction performance than those proposed by Jiang and Zhao [112], Jiang [113], and Cai *et al.* [115]. The strength prediction errors for the six rock types and two *in situ* rock masses are 2.0724%–3.5091% and 1.0144%–3.2321%, respectively. This criterion shares the same parameter system with the GHB criterion and fully inherits the parameter advantages.

5. Conclusions and outlook

(1) Intercrossing and fusing the GNST with metaheuristic algorithms and establishing an open and shared information service platform using modern media technology are hot topics and will be the future research directions.

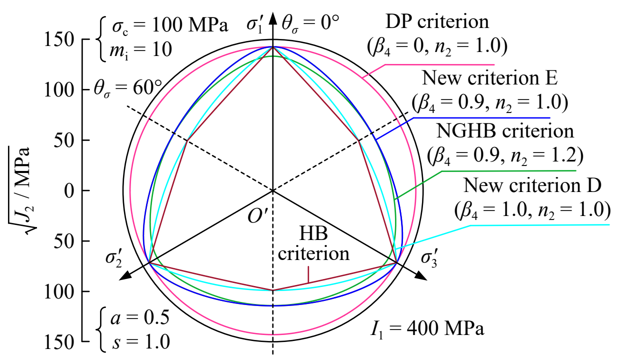

Fig. 20. Influence of n_2 and β_4 on the failure envelope of the NGHB criterion [17].

With its continuous improvement, modification, and perfection, the nonlinear strength theory has been unified, making each single nonlinear strength theory applicable to specific materials become the GNSTs applicable to all materials and forming a complete strength theory system. However, most GNSTs are based on classical criteria or empirical methods to determine material parameters. The deviatoric plane shape of the GNST cannot show all smooth and convex curves well. Therefore, optimizing material parameters based on metaheuristic algorithms will be the research direction of geomaterial strength.

(2) Constructing a set of strength index systems of *in-situ* engineering rock mass considering the directionality of the rock mass structural plane, *in-situ* regionality, and size effect to obtain dynamic *in-situ* rock mass parameters quickly, conveniently, and accurately will become the research direction for the basic theoretical rock mechanics.

It is required to research intelligent equipment for the efficient acquisition of *in-situ* rock mass parameters to develop portable intelligent collection equipment for *in-situ* rock mass data of underground engineering, to process real-time data and topology structure–recognition algorithms, to construct an evaluation index system for the rock mass quality of underground engineering based on multisource information fusion under complex occurrence conditions, to acquire the dynamic *in-situ* rock mass parameters quickly, easily, and accurately using the new generation communication networks (such as the 6G network, Internet of Things, and mobile communication), and modern information means (such as cloud computing), and finally to realize standardization and intellectualization.

(3) Proposing digital, informative, and intelligent *in-situ* strength construction methods for underground engineering rock is essential for solving the bottleneck problem.

Regarding the *in-situ* engineering rock mass strength under complex occurrence conditions, it is urgent to develop large-scale 3D numerical software to seamlessly integrate measures across sample scale, *in-situ* test scale, and engineering scale. This software should coordinate theory, test, simulation, and other methods to establish a new generation of digital, informative, and intelligent construction methods for assessing *in-situ* rock mass strength in underground engineering. Such practice would become a research idea and solution, break through the key scientific bottleneck problems in underground engineering, and eventually realize the construction of *in-situ* rock mass strength in full intelligence.

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Conflict of Interest

Shunchuan Wu is an editorial board member of this journal and was not involved in the editorial review or the decision to publish this article. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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