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DIRECTED HYPERGRAPH THEORY AND DECOMPOSITION CONTRACTION METHOD

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ABSTRACT A new branch of hypergraph theory-directed hypergraph theory and a kind of new methods-dicomposition contraction(DCP, PDCP and GDC) methods are presented for solving hypernetwork problems. Its computing time is lower than that of ECP method in several order of magnitude.

KEY WORDS directed hypergraph theory, decomposition contraction method, hypernetwork

THIS paper presents a directed hypergraph theory (DHT), as the extension of the hypergraph theory in Ref. [1], for solving directed (active) hypernetwork problems. Then, one application of DHT is proposed. That is, by introducing the concept of decomposition contraction pair(DCP), ECP method [2] is improved into a DCP method which is more efficient and concise. The computing time of DCP is lower than that of ECP in several orders of magnitude. At last, DCP method is developed into a PDCP method for a cascade of two active networks, and into a GDC method for the interconnection of $e>2$ multiterminal active networks(MAN).

1 DIRECTED HYPERGRAPH THEORY

1.1 HYPERNETWORKS AND DIRECTED HYPERGRAPHS

A hypernetwork N is the interconnection of $e \geqslant 2$ MAN called subnetworks. A subnetwork N_j is called an m_j-terminal subnetwork if it has m_j accessible vertices, and called terminals, for connection to other subnetworks. Vertices of N_j other than its terminals are called internal vertices of N_j. The terminal set and internal vertex set of N_j are denoted by E_j and X_j', respectively. The composite graph of N is denoted as $G=(X, U)$ [3, 4], as shown is Fig. 1.

The composite graph G_j of the indefinite-admittance matrix Y_j of N_j is called an m_j-terminal composite graph, and denoted by

$$G_j=(X_j, U_j)=(E_j \bigcup X_j', U_j), \ E_j \bigcap X_j'=\Phi, \ |E_j|=e_j \quad (1)$$

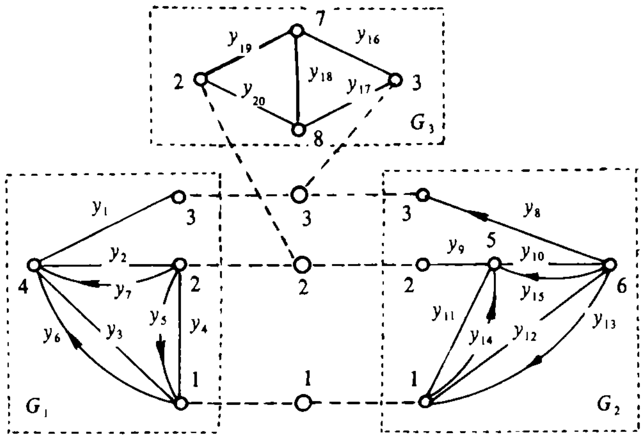

Fig. 1 The composite graph G of a hypernetwork N

where U_j is the edge set of G_j consisting of directed and undirected edges. E_j is also referred to as a hyperedge, and G_j the associated composite graph of E_j, denoted by $G_j = G(E_j)$. If N_j is passive, then G_j and E_j are undirected, the terminals of E_j have not assigned any polarity. If N_j is active, then G_j contains directed edges, each terminal of E_j must be assigned a polarity for some problem. If one terminal of E_j is chosen as positive- (negative-) pole called positive- (negative-) root, others as negative- (positive-) poles, then E_j is called a positive- (negative-) rooted hyperedge or directed hyperedge. The geometrical representation of hyperedges are shown in Fig. 2(a) , in which E_1 and E_2 are positive- rooted, and E_3 is undirected. We draw out one line called lead from each terminal of E_j for connecting to other hyperedges conveniently, and it is considered not as an edege but as a short-circuit line. The set of all leads of E_j is called lead set L_j.

A directed hypergraph H is an ordered pair (V, E) consisting of a terminal set V and hyperedge set E containing directed and undirected hyperedges, as shown in Fig. 2(a), that is ,

$$H = (V, E), \quad V = \{1, 2, \cdots, v\}, \quad E = \{E_j \mid j = 1, \cdots, e\} \tag{2}$$

where $E_j = E_j[v_1, \cdots, v_{e_j}] = \{v_1, \cdots, v_{e_j}\} \subset V$, $E_j \neq \Phi$, $\cup_{j=1}^{e} E_j = V$

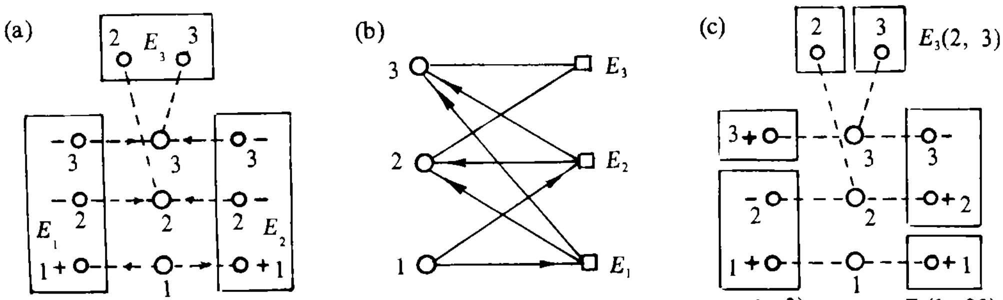

Fig. 2 Directed hypergraph and its corresponding bipartite graph

(a) hypergraph H; (b) corresponding bipartite graph $G<H>$; (c) a hypertree T_1

H is undirected if it contains undirected hyperedges only. In this paper, we only consider undirected and positive-rooted hypergraphs. The composite graph $G = U_{j=1}^{e} G_j$ of N is called the associated composite graph of H, and denoted as $G = G(H)$.

1.2 OPERATIONS ON HYPEREDGES

We introduce a few operations involving hyperedges. Suppose $E_j = A_j \cup B_j$, $A_j \cap B_j = \Phi$, denote this E_j by $E_j[A_j B_j]$. A_j is said to be opencircuited if all leads of A_j are removed from E_j and E_j becomes $E_j[B_j] = E_j - A_j$, or contracted if A_j and its associated graph $G(A_j)$ are contracted into one terminal (called hyperterminal) $\bar{A}_j$ and E_j becomes a $|E_j - A_j| + 1$ terminal hyperedge, denoted by $E_j[\bar{A}_j B_j] = E_j \bigcirc A_j$

If each E_j of $H = (V, E)$ is contracted to form a vertex, still denoted by E_j, and the lead set L of H is considered as an edge set consisting of directed and undirected edges, the resulted composite graph is called the corresponding bipartite graph of H, denoted by $G<H> = (V, E; L)$, as shown in Fig. 2(b).

An h-decomposition $D_j(m, h) = E_j[F_1, \cdots, F_h]$ of an m-terminal undirected hyperedge E_j is a partition of E_j into h nonempty subsets F_i, $i = 1, \cdots, h$, called undirected subhyperedges, that is,

$$E_j[F_1, \cdots, F_h] = \{F_1, \cdots, F_h\}, \qquad E_j = \mathrm{U}_{i=1}^{h} F_i, \qquad F_i \cap F_k = \Phi, \qquad i \neq k \tag{3}$$

where F_i is the i-th undirected subhyperedge of E_j with its terminal juxtaposition as its simplified expression. For example, $E_j[123] = \{1, 2, 3\}$ has a 2-decomposition $E_j[12, 3] = \{12, 3\}$. The number of h-decompositions of an m-terminal undirected hyperedge is denoted by $[m, h]$, called the stirling number of the second kind in combinatorial theory [5]. The weight $D_j(y)$ of D_j is defined by a polynomial $P_j(F_1, \cdots, F_h)$ formed from the sum of all h-tree weights $t_{F_1, \cdots, F_h}(y)$ in the associated graph $G(E_j)$ of E_j, i. e. ,

$$D_j(y) = P_j(F_1, \cdots, F_h) = \Sigma t_{F_1, \cdots, F_h}(y) | G(E_j) \tag{4}$$

Theorem 1 (Recursive Formula for $SD(m)$) Let the decomposition set $SD(m)$ of an m-terminal undirected hyperedge $E[1 \cdots m] = \{1, \cdots, m\}$ be

$$\begin{gathered} SD(m) = \mathrm{U}_{h=1}^{m} SD(m, h) \\ SD(m, h) = \{D(m, h, a) | a = 1, \cdots, [m, h]\} \\ D(m, h, a) = \{F(m, h, a, b) | b = 1, \cdots, h\} \end{gathered} \tag{5}$$

where $SD(m, h)$ and $D(m, h, a)$ are the h-decomposition set and the a-th h-decomposition of $E\{1 \cdots m\}$, respectively, and $F(m, h, a, b)$ is the b-th subhyperedge of $D(m, h, a)$. Then the decomposition set $SD(m+1)$ of $(m+1)$-terminal undirected hyperedge $E[1 \cdots m+1] = \{1, \cdots, m+1\}$ is

$$\begin{gathered} SD(m+1) = \mathrm{U}_{h=1}^{m+1} SD(m+1, h) \\ SD(m+1, h) = \{D(m+1, h, a) | a = 1, \cdots, [m+1, h]\} \end{gathered} \tag{6}$$

$$D(m+1, h, a) = \begin{cases} \{F(m, h-1, a, 1), \cdots, F(m, h-1, a, h-1), m+1\}, \ a = 1, \cdots, [m, h-1] \\ \{F(m, h, d, 1), \cdots, F(m, h, d, b) \cup \{m+1\}, \cdots, F(m, h, d, h)\}, \end{cases}$$

$$a = [m, h-1] + (d-1)h + b; \ d = 1, \cdots, [m, h]; \quad b = 1, \cdots, h,$$

and the cardinality of $SD(m+1, h)$ is

$$|SD(m+1,\ h)| = [m+1,\ h] + h[m,\ h] \tag{7}$$

This theorem can be proved easily according to the definition of $D(m, h, a)$ and Ref. [5].

Corollary 1.1 By using the recursive formula (6), starting from $SD(1)=\{\{1\}\}$, we can obtain

$$SD(2)=\{\{12\},\ \{1,\ 2\}\},$$

$$SD(3)=\{\{123\},\ \{12,\ 3\},\ \{13,\ 2\},\ \{1,\ 23\},\ \{1,\ 2,\ 3,\}\},$$

$$SD(4)=\{\{1234\},\ \{123,\ 4\},\ \{124,\ 3\},\ \{12,\ 34\},\ \{134,\ 2\},\ \{13,\ 24\},\ \{14,\ 23\},\ \{1,\ 234\},\ \{12,\ 3,\ 4\},\ \{13,\ 2,\ 4\},\ \{1,\ 23,\ 4\},\ \{14,\ 2,\ 3\},\ \{1,\ 24,\ 3\},\ \{1,\ 2,\ 34\},\ \{1,\ 2,\ 3,\ 4\}\}$$

Corollary 1.2 The total number of decompositions $|SD(m)|$ of $E[1\cdots m)$ is equal to the Bell number [5] $B_m=[m,\ 1]+\cdots+[m,\ m]$. Especially, $B_1=1$, $B_2=2$, $B_3=5$, $B_4=15$, $B_5=52$, and so on.

A positive-rooted h-decomposition $PD_j(m,\ h)_r$, with a root set $r=\{r,\ \cdots,\ r_h\}$, of an m-terminal directed hyperedge E_j is a partition of E_j into h nonempty positive-rooted subhyperedges r_iw_i, with a positive root r_i, $i=1,\ \cdots,\ h$, denoted by

$$PD_j(m,h)_r = E_j(rw) = E_j(r_1w_1,\ \cdots,\ r_hw_h) = \{r_1w_1,\ \cdots,\ r_hw_h\} \tag{8}$$

where $rw=r_1w_1,\ \cdots,\ r_hw_h$. The weight $PD_j(y)$ of PD_j is defined by a polynomial $P_j(r_1w_1,\ \cdots,\ r_hw_h)$ formed from the sum of all positive-rooted h-tree weights $t_{r_1w_1,\cdots,r_hw_h}(y)$ in the associated directed graph $G(E_j)$ of E_j, that is

$$PD_j(y) = P_j(rw) = P_j(r_1w_1,\ \cdots,\ r_hw_h) = \Sigma t_{r_1w_1,\cdots,r_hw_h}(y)|_{G(E_j)} \tag{9}$$

The positive-rooted decomposition set $SPD(m)_p$, with a given positive-root set $p=\{p_1,\ \cdots,\ p_h\}$, of an m-terminal directed hyperedge $E[1\cdots m]$ is the set of all its positive-rooted decompositions $PD(m,\ h,\ c)_r$ with a root set $r\supseteq p$, i. e. ,

$$SPD(m)_p=\{PD(m,\ h,\ c)_r|r\supset p;\ h=|p|,\ \cdots,\ m;\ c=1,\ \cdots,\ [m,\ h]_p\} \tag{10}$$

where $r=\{r_1,\ \cdots,\ r_h\}=\{p_1,\ \cdots,\ p_n,\ r_{n+l},\ \cdots,\ r_h\}$, and $[m,\ h]_p$ is the number of positive-rooted h-decompositions $PD(m,\ h,\ c)_r$ with a given positive-root set p of $E[1\cdots m]$. For example, $SPD(3)_1=\{123\},\ \{12,\ 3\},\ \{13,\ 2\},\ \{1,\ 23\},\ \{1,\ 32\},\ \{1,\ 2,\ 3\}\}$, $SPD(3)_{2,3}=\{\{21,\ 3\},\ \{2,\ 31\},\ \{2,\ 3,\ 1\}\}$. From the definition of $SPD(m)_p$ and Corollary 1.1 , we can obtain the following corollaries.

Corollary 1.3 (Algorithm for Finding $SPD(m)_p$) $SPD(m)_p$ can be found by the following steps using pseudo-SPARKS language [6] :

(1) $p\leftarrow\{p_1,\ \cdots,\ p_n\}$; $n\leftarrow|p|$; $SPD(m)_p\leftarrow\Phi$; input m and $SD(m)$

(2) for $h\leftarrow n$ to m do $c\leftarrow 0$; $SPD(m,\ h)\leftarrow\Phi$

for $a\leftarrow 1$ to $[m,\ h]$ do from $SD(m)$ take out $D(m,\ h,\ a)$;

if $D(m,\ h,\ a)=\{p_1w_1,\ \cdots,\ p_nw_n,\ F_{n+1},\ \cdots,\ F_h\}$, then

for $i_{n+1}\leftarrow 1$ to $|F_{n+1}|$ do $r_{n+1}\leftarrow F_{n+1}(i_{n+1})$; $w_{n+1}\leftarrow F_{n+1}-F_{n+1}(i_{n+1})$

$\vdots$

for $i_h\leftarrow 1$ to $|F_h|$ do $r_h\leftarrow F_h(i_h)$; $w_h\leftarrow F_h-F_h(i_h)$; $c\leftarrow c+1$;

$PD(m,\ h,\ c)\leftarrow\{p_1w_1,\ \cdots,\ p_nw_n,\ r_{n+1}w_{n+1},\ \cdots,\ r_hw_h\}$;

$SPD(m, h) \leftarrow SPD(m, h) \cup \{PD(m, h, c)\}$

repeat

⋮

repeat

end if

repeat

$SPD(m)_p \leftarrow SPD(m)_p \cup SPD(m, h)$

repeat

Corollary 1.4 let $SPD(m)=SPD(m)_1$. From Corollaries 1.1 and 1.3 we can obtain

$$SPD(1)=\{\{1\}\}$$

$$SPD(2)=\{\{12\}, \{1, 2\}\},$$

$$SPD(3)=\{\{123\}, \{12, 3\}, \{13, 2\}, \{1, 23\}. \{1. 32\}, \{1, 2, 3\}\},$$

$$\begin{aligned}SPD(4)=\{&\{1234\}, \{123, 4\}, \{124, 3\}, \{12, 34\}, \{12, 43\}, \{134, 2\}, \{13, 24\}, \{13, 42\},\\ &\{14, 23\}, \{14, 32\}, \{1, 234\}, \{1, 324\}, \{1, 423\}, \{12, 3, 4\}, \{13, 2, 4\},\\ &\{1, 23, 4\}, \{1, 32, 4\}, \{14, 2, 3\}, \{1, 24, 3\}, \{1, 42, 3\}, \{1, 2, 34\}, \{1, 2, 43\},\\ &\{1, 2, 3, 4\}\end{aligned}$$

⋮

1.3 DECOMPOSED HYPERGRAPHS AND DIRECTED HYPERTREES

A positive-rooted decomposed hypergraph $H_d=(V, F)$ of a hypergraph $H=(V, E)$, $E=\{E_j|j=1, \cdots, e\}$, is a hypergraph formed by taking one positive-rooted decomposition PD_j from each directed hyperedge E_j and one decomposition D_k from each undirected hyperedge E_k of H such that their union forms a subhyperedge set $F=\{F_i|i=1, \cdots, f\}$. H_d is simple if there is at most one common terminal between any two subhyperedges of H_d . H_d is undirected if H is undirected. The corresponding bipartite graph of H_d is denoted by $G<H_d>=(V, F; L)$.

A hyperpath P of length q of H_d is a sequence

$$P(v_1-v_{q+1})=[v_1, F_1', v_2, F_2', \cdots, v_q, F_q', v_{q+1}] \tag{11}$$

satisfying the following conditions :

(1) $v_1, v_2, \cdots, v_{q+1}$ are all distinct terminals of H_d.

(2) $F_i'=\{v_i, v_{i+1}\}$, $i=1, \cdots, q$, are all distinct subhyperedges, and each F_i' is obtained from such F_i by open-circuiting all the terminals of F_i besides v_i and v_{i+1} that $F_i \cap F_{i+1}=\{v_{i+1}\}$, $i=1, \cdots, q-1$, and there is no other common terminal between F_i and F_j , $i\neq j$, $i, j\in\{1, \cdots, q\}$.

P is directed if for each directed F_i' of P, v_i and v_{i+1} are the positive- and the negative-poles of F_i' respectively, and denoted by

$$P(v_1 \rightarrow v_{q+1}) = (v_1, F_1', v_2, F_2', \cdots, v_q, F_q', v_{q+1}) \tag{12}$$

P is undirected if all F_i', $i=1, \cdots, q$, are undirected. An undirected hyperpath P can be considered as either $P(v_1 \rightarrow v_{q+1})$ or $P(v_{q+1} \rightarrow v_1)$.

If $q>1$ and $v_{q+1}=v_1$, $P(v_1-v_{q+1})$ and $P(v_1 \rightarrow v_{q+1})$ are called hypercircuit $C(v_1-v_q)$ and directed hypercircuit $C(v_1 \rightarrow v_q)$ respectively.

Two terminals v_i and v_j are (directed) connected, denoted by $v_i - v_j$ ($v_i \rightarrow v_j$), in H_d if there exists a (directed) hyperpath from v_i to v_j in H_d. They are strongly connected, denoted by $v_i \rightleftarrows v_j$, if $v_i \rightarrow v_j$ and $v_j \rightarrow v_i$, H_d is (strongly) connected if all its terminals are (strongly) connected one with another. The relation $v_i - v_j$ ($v_i \rightarrow v_j$) is an equivalence relation, whose classes are called the (strongly) connected components of H_d. It is easily to show the following theorem.

Theorem 2 A terminal-hyperedge-interchange sequence P of H_d is a (directed) hyperpath of length q if and only if its corresponding bipartite graph $G<P>$ is a (directed) path of length $2q$.

Corollary 2.1 H_d has k (strongly) connected components if and only if $G<H_d>$ has k (strongly) connected components.

Let $p = \{p_1, \cdots, p_k\} \subset V$; $pq = p_1q_1, \cdots, p_kq_k$; $q_1, \cdots, q_k \subset V$. A positive-rooted k-hypertree T_{pq}, or simply called directed k-hypertree, of a directed hypergraph $H = (V, E)$, $E = \{E_j | j = 1, \cdots, e\}$, is a simple positive-rooted decomposed hypergraph H_d of H that contains no hypercircuit, whose connected-component number is k, and whose i-th component $T_{p_iq_i}$ contains positive root p_i and terminal subset q_i of V and has a unique directed hyperpath $P(p_i \rightarrow v_j)$ from p_i to each terminal $v_j (\neq p_i)$ of $T_{p_iq_i}$. T_{pq} is undirected if H is undirected. A directed 1-hypertree is simply called a direted hypertree. For convenience, T_{pq} is expressed by the product of all hyperedge decompositions $E_j(rw)$, $j = 1, \cdots, e$, contained in it, that is,

$$T_{pq} = T_{p_1q_1, \cdots, p_kq_k} = \prod_{j=1}^{e} E_j(rw) \tag{13}$$

For example, $T_1 = E_1(12, 3)E_2(1, 23)E_3(2, 3)$ is shown in Fig. 2(c). The weight of T_{pq} is denoted by $T_{pq}(y)$ and defined as follows :

$$T_{pq}(y) = \prod_{j=1}^{e} E_j(rw, y) = \prod_{j=1}^{e} P_j(rw) \tag{14}$$

From the definition of T_{pq} and Theorem 2 we have the following theorem.

Theorem 3 A positive-rooted decomposed hypergraph $H_d = (V, F)$ of a directed hypergraph H is a positive-rooted k-hypertree T_{pq} if and only if its corresponding bipartite graph $G<H_d> = (V, F; L)$ is a positive-rooted k-tree [3] t_{pq}.

1.4 GENERATIN $P_G(t_1(y))$ BY $P_H(T_1(y))$

One application of DHT is to generate $P_G(t_1(y))$ by $P_H(T_1(y))$.

Theorem 4 Let $H = (V, E)$, $E = \{E_j | j = 1, \cdots, e\}$, be a directed hypergraph, $G = G(H)$ and $G_j = G(E_j)$ be the associated composite graphs of H and E_j , $j = 1, \cdots, e$, respectively. Then the positive-rooted k-tree weight polynomial [3] of G , $P_G(t_{pq}(y)) = \Sigma_G t_{pq}(y)$, can be generated by the positive-rooted k-hypertree weight polynomial of H, $P_H(T_{pq}(y)) = \Sigma_H T_{pq}(y)$, as follows:

$$P_G(t_{pq}(y)) = P_H(T_{pq}(y)) = \Sigma_H \Pi_{j=1}^{e} P_j(rw) \tag{15}$$

where $P_j(rw)$ can be generated by Algorithm DKTPCG in Ref. [7]. Especially, for $k = 1$ and $pq = 1$, we have

$$P_G(t_1(y)) = P_H(T_1(y)) = \Sigma_H T_1(y) \tag{16}$$

This theorem can be easily proved by showing that any term of the right-side of Eq. (15) is a t_{pq} of G and any t_{pq} of G can be expressed as one term of the right-side of Eq. (15).

2 DECOMPOSITION CONTRACTION METHODS

For applying Eq. (16) , the key problem is how to generate $P_H(T_l(y))$ more efficiently. We first discuss the following two special methods: DCP and PDCP methods.

2.1 DCP METHOD

Consider an undirected 2-hyperedge hypergraph $H=(V, E)$, $v=\{1, \cdots, m\}$, $E=\{E_1, E_2\}$, a pair of decomposition hyperedges $D_1=E_1[F_1, \cdots, F_h]$ and $D_2=E_2[F_1', \cdots, F_k']$ $(h+k=m+1)$ are said to be a pair of essential complementary partition (ECP) of V, denoted by $ECP=(D_1, D_2)$, if they form a hypertree T of H. A decomposition hyperedge $D_1=E_1[F_1, \cdots, F_h]$ and its corresponding contraction hyperedge $C_2=E_2[\overline{F_1\cdots F_h}]$ are called a decomposition-contraction pair (DCP) of H, and denoted by $DCP=(D_1, C_2)$. Its weight $DCP(y)=D_1(y)C_2(y)$.

Theorem 5 The hypertree weight polynomial $P_h(T(y))$ of an undirected 2-hyperedge hypergraph H is equal to the sum of all decomposition-contraction pair weights $DCP(y)$'s of H, that is ,

$$P_H(T(y))=\sum DCP(y)=\sum D_1(y)C_2(y) \tag{17}$$

where the summations are taken over all decomposition hyperedges D_1's of E_1.

Proof Suppose there are b D_2's essential comlementary with some D_1, denoted by D_2^i, $i=1, \cdots, b$, then the part of $P_H(T(y))$ containing D_1 is $P_H(T(y)|D_1)=D_1(y)\Sigma_{i=1}^{b}D_2^i(y)$. Since terminals of F_j have been connected by D_1, they must be disconneted in each D_2^i by the definition of hypertree. After contracting F_j , $j=1, \cdots, h$, all D_2^i will become the same C_2, hence $\Sigma_{i=1}^{b}D_2^i(y)=C_2(y)$, and

$$P_H(T(y))=\Sigma P_H(T(y)|D_1)=\Sigma D_1(y)\Sigma_{i=1}^{b}D_2^i(y) \qquad \text{(ECP Method)} \tag{18}$$

$$=\Sigma D_1(y)C_2(y)=\Sigma DCP(y) \qquad \text{(DCP Method)} \tag{19}$$

For example , the ECP's and DCP for $m=4$ and $D_1=E_1[1, 23, 4]$ are shown in Fig. 3. The term numbers of $P_H(T(y))$ for DCP and ECP Methods are Bm (Bell number) and $Nm=2(m+1)^{m-2}$ respectively. $B_2=2$, $B_3=5$, $B_4=15$, $B_5=52$, $B_6=203$, $B_7=877$, $\cdots$, $N_2=2$, $N_3=8$, $N_4=50$, $N_5=432$, $N_6=4802$, $N_7=65536$, $\cdots$. When m increasing, Bm can be lower than Nm in several orders of magnitude, hence DCP Method is more concise and efficient than ECP Method.

2.2 PDCP METHOD

If N is an active hypernetwork, its associated hypergraph H will be directed. The concept of Chen's ECP should be developed as follows. A pair of positive-rooted decomposition hyperedges $PD_1=E_1(r_1w_1, \cdots, r_hw_h)$ and $PD_2=E_2(r_1'w_1', \cdots, r_k'w_k')(h+k=m+1)$, $r_1=r_1'=1)$ are said to be a pair of positive-rooted essential complementary partition (PECP) of V, denoted by $PECP=(PD_1, PD_2)$, if they form a positive-rooted hypertree T_1 of H. A positive-rooted decomposition hyperedge $PD_1=E_1(r_1w_1, \cdots, r_hw_h)$

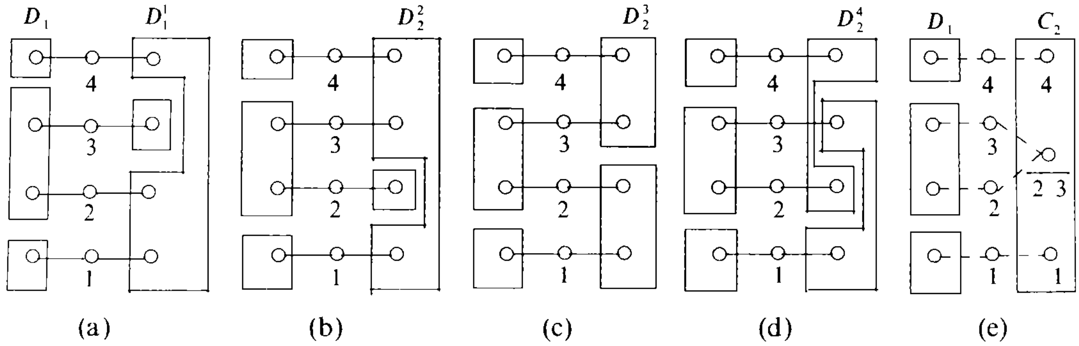

Fig. 3 ECP's and DCP for $m=4$ and $D_1=E_1[1, 23, 4]$
(a) $D_2^1=E_2[124, 3]$; (b) $D_2^2=E_2[134, 2]$; (c) $D_2^3=E_2[12, 34]$; (d) $D_2^4=E_2[13, 24]$; (e) $C_2=E_2[\overline{1234}]$

and its corresponding positive-rooted contraction hyperedge $PC_2=E_2(\overline{r_1w_1}^+\cdots\overline{r_hw_h}^+)$ are called a positive-rooted decomposition-contraction pair (PDCP) of H , and denoted by $PDCP=(PD_1, PC_2)$, where the right superscript "+" of w_j^+, $j=1, \cdots, h$, mean that before contracting r_jw_j all edges incident into the vertices of w_j in $G(E_2)$ must deleted to ensure all vertices of w_j being positive poles in PD_2 (since they are negative poles in PD_1). Its weight $PDCP(y) = PD_1(y)PC_2(y)$.

Theorem 6 The positive-rooted hypertree weight polynomial $P_H(T_1(y))$ of a directed 2-hyperedge hypergraph H is equal to the sum of all positive-rooted decomposition-contraction pair weights $PDCP(y)$'s of H, that is ,

$$P_H(T_1(y))=\Sigma PDCP(y)=\Sigma PD_1(y)PC_2(y) \tag{20}$$

where the summations are taken over all PD_1's of E_1.

Proof Suppose there are a $(a\leqslant b)$ PD_2's positive-rooted essential complementary with some PD_1, denoted by PD_2^i, $i=1, \cdots, a$, then we have $\Sigma_{i=1}^{a}PD_2^i(y)=PC_2(y)$, hence

$$P_H(T_1(y))=\Sigma P_H(T_1(y)|PD_1)=\Sigma PD_1(y)\Sigma_{i=1}^{a}PD_2^i(y) \qquad \text{(PECP Method)} \tag{21}$$

$$=\Sigma PD_1(y)PC_2(y)=\Sigma PDCP(y) \qquad \text{(PDCP Method)} \tag{22}$$

For example , the PECP's and PDCP for $m=4$ and $PD_1=E_1(1, 23, 4)$ are shown in Fig. 4, where $PD_2^1(y)+PD_2^2(y)=PC_2(y)$. The analysis in Section 3 of Ref. [8] is PECP Method, here it is improved into the more concise and efficient PDCP Method.

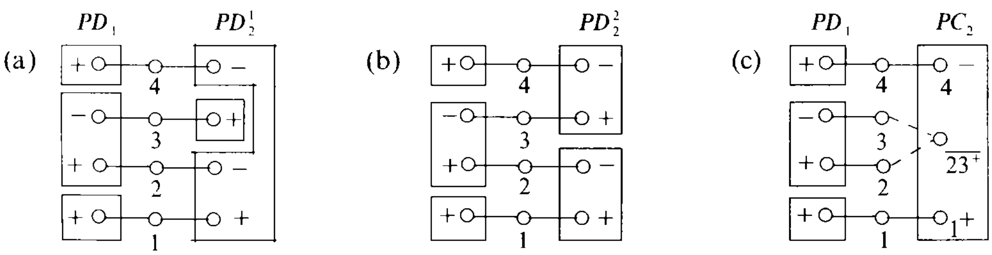

Fig. 4 PECP's and PDCP for $m=4$ and $PD_1=E_1(1, 23, 4)$
(a) $PD_2^1=E_2(124, 3)$; (b) $PD_2^2=E_2(12, 34)$; (c)$PC_2=E_2(1\,\overline{23}^+4)$

2.3 GDC METHOD

Now we extend PDCP analysis to the general directed hypergraph with $e>2$. In this case, after processing each hyperedge, all other hyperedges not processed must be contracted correspondingly. Hence we have

Threorem 7 The positive-rooted hypertree weight polynomial $P_H(T_1(y))$ of a directed hypergraph H, $e>2$, can be expressed by

ergraph H, $e>2$, can be expressed by

$$P_H(T_1(y))=\Sigma PD_1(y)\cdots\Sigma PC_jD_j(y)\cdots PC_e(y) \tag{23}$$

where PC_jD_j, $1<j<e$, called the $(j-1)$ -order positive-rooted contraction-decomposition hyperedge [$(j-1)$-order PCD] of E_j, is obtained from E_j by taking $(j-1)$-order contraction according to preceding $(j-1)$ processed hyperedges and a succeeding positive-rooted decomposition not destroying the connectedness of the resulting hypergraph; PC_e, called the $(e-1)$-order positive-rooted contraction hyperedge [$(e-1)$-order PC] of Ee, is obtained from Ee by taking $(e-1)$-order positive-rooted contraction according to preceding $(e-1)$ processed hyperedges; and the summations are taken over all such PD_1 of E_1 and PC_jD_j of E_j, $1<j<$e, that not destroying the connectedness of the resulting hypergraphs. If E_1 (E_j or E_e)is undirected, PD_1 (PC_jD_j or PC_e) can be replaced by D_1(C_jD_j or C_e).

Example Given a composite graph $G=(X, U)$ of a hypernetwork N, $X=\{1, \cdots, 8\}$, $U=\{y_1, \cdots, y_{20}\}$, as shown in Fig. 1. its associated directed hypergraph $H=(V, E)$, $V=\{1, 2, 3\}$, $E=\{E_1, E_2, E_3\}$, $E_1=\{1, 2, 3\}$ and $E_2=\{1, 2, 3\}$ are directed, $E_3=\{2, 3\}$ is undirected, as shown in Fig. 2(a). Find $P_H(T_1(y))$ of H.

Solution Let $PC_jD_j(y)=P_j(rw)$ for $PC_jD_j=E_j(rw)$, and so on. By applying Theorem 7, we can obtain

$$\begin{aligned}P_h(T_1(y))=&P_1(123)P_2(\overline{123})P_3(\overline{23})+P_1(12,\ 3)[P_2(\overline{12}3)P_3(\overline{23})+P_2(\overline{12},\ 3)P_3(23)]\\&+P_1(13,\ 2)[P_2(\overline{13}2)P_3(\overline{23})+P_2(\overline{13},\ 2)P_3(32)]+P_1(1,\ 23)P_2(\overline{123^+})P_3(\overline{23})\\&+P_1(1,\ 32)P_2(\overline{132^+})P_3(\overline{32})+P_1(1,\ 2,\ 3)[P_2(123)P_3(\overline{23}))\\&+P_2(12,\ 3)P_3(23)+P_2(13,\ 2)P_3(32)]\end{aligned}$$

Note that $P_2(\overline{12^+3^+})=P_2(\overline{123})$, $P_3(\overline{2^+3^+})=P_3(\overline{23})$, $P_2(\overline{12^+}3)=P_2(\overline{12}3)$, and so on. $P_1(123)$, $P_2(\overline{123})$ and $P_3(\overline{23})$, etc., can be found by Algorithm DKTPCG in Ref. [7].

3 CONCLUSIONS

(1) The directed hypergraph theory is a new extension of the undirected hypergraph theory in Ref. [1].

(2) The decomposition contraction method is a powerful hypergraph theory method. It is more efficient, concise and extensive than ECP method.

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有向超图论和分解收缩法

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摘 要 提出了超图理论的一个新分支－有向超图理论和一类新方法－分解收缩(DCP, PDCP 和 GDC)法，用于示解超网络问题．它计算时间比 ECP 法降低几个数量级．

关键词 有向超图理论，分解收缩法，超网络