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Free boundary value problems for a class of generalized diffusion equation

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Abstract: The transport behavior of free boundary value problems for a class of generalized diffusion equations was studied. Suitable similarity transformations were used to convert the problems into a class of singular nonlinear two-point boundary value problems and similarity solutions were numerical presented for different representations of heat conduction function, convection function, heat flux function, and power law parameters by utilizing the shooting technique. The results revealed the flux transfer mechanism and the character as well as the effects of parameters on the solutions.

Key words: free boundary value problem; generalized diffusion equation; nonlinear boundary problem; shooting technique

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1 Introduction

In this paper, we study the transfer behavior of free boundary value problem for a class of generalized diffusion equation

$$u_t = (k(u)|u_x|^{n-1}u_x)_x + t^{-\frac{n}{n+1}}h(u)u_x \quad \text{for} \quad (x,t)\in G \tag{1}$$

$$u|_{x=\mu(t)} = 0 \quad \text{for} \quad t>0 \tag{2}$$

$$(k(u)|u_x|^{n-1}u_x)|_{x=\mu(t)} = \alpha\mu'(t) \tag{3}$$

$$u|_{t=0} = \beta \qquad \text{for} \quad x>\mu(0)=0 \tag{4}$$

where $k(u)$ and $h(u)$ are assumed to be real-continuous differential functions defined on $[0,\beta]$, $k(u)>0$ for $u>0$. $G \overset{\text{def}}{=} \{(x,t); x>\mu(t) \text{ and } t>0\}$, $\mu(t)$ is an unknown function and must be determined as a part of the solution. $N>0$, $\alpha\le 0$, and $\beta>0$ are given constants. Equation (1) was firstly suggested as a model for certain generalized diffusion processes by Philip [1].

Recently, Wang [2] and Zheng [4] have considered some generalized diffusion equations similar to equation (1) at some certain initial, boundary conditions. Existence, uniqueness results were analytically established by employing the similarity transformation and perturbation technique. But the behavior of the solutions of equation (1) with free boundary conditions (2)-(4) are not understood at the present time.

Clearly, when $n\neq 1$, the generalized diffusion equation (1) appeared a super-nonlinear because its term $(k(u)|u_x|^{n-1}u_x)_x$ ($n>0$, $n\neq 1$) and having a free boundary, it is too difficult to study directly. Therefore, special emphasis is given to the formulation of the generalized diffusion equations which provide similarity solutions.

2 Two-point boundary value problem

We look for the similarity solutions of the form

$$u(x,t)=\phi(\eta), \quad \eta = x/t^{1/(n+1)} = x\cdot t^{-\frac{1}{n+1}}, \ \mu(t)=\eta_0 t^{1/(n+1)} \tag{5}$$

Substituting (5) into equations (1)-(4), we arrive the following nonlinear boundary value problem:

$$(k(\phi(\eta))|\phi'(\eta)|^{n-1}\phi'(\eta))' + (\frac{\eta}{(n+1)} + h(\phi(\eta)))\phi'(\eta),$$

$$\eta>\eta_0 \tag{6}$$

$$\phi(\eta_0)=0, \quad \phi(+\infty)=\beta>0 \tag{7}$$

$$(k(\phi(\eta))|\phi'(\eta)|^{n-1}\phi'(\eta))|_{\eta=\eta_0} = \frac{\alpha}{n+1}\eta_0 \tag{8}$$

Conversely, if $\phi(\eta)$ is a solution to equations (6)-(8), then the function $(u(x;t),\mu(t))$ must be a solution to generalized diffusion equations (1)-(4). Therefore, in what follows, we shall only pay our attention to the problems (6)-(8).

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Let $s=\phi(\eta)$ be a solution to the nonlinear boundary value problems (6)-(8). If $\phi(\eta)$ is strictly increasing in $[\eta_0,+\infty)$, then $\phi'(+\infty)=0$, and the function $\eta=y(s)$ inverse to $s=\phi(\eta)$ exists. And $s=\phi(y(s))$ on $(0,\ \beta)$, $\eta=y(\phi(\eta))$ on $[\eta_0,+\infty)$, $\phi'(y(s))=1/y'(s)$ holds in $(0,\ \beta)$. Substituting $\eta=y(s)$ into equations (6)-(8), note that $|\phi'(\eta)|^{n-1}\phi'(\eta)=(\phi'(\eta))^n$ and putting $\theta(s)=\dfrac{k(s)}{(y'(s))^n}$ (where $\theta(s)>0$ is diffusion flux, $0<s<\beta$), then we formally obtain the following singular nonlinear two-point boundary value problems:

$$\theta''(s)=-\frac{1}{n+1}\left(\frac{k(s)}{\theta(s)}\right)^{1/n}+h'(s),0<s<\beta \tag{9}$$

$$\alpha\cdot\theta'(0)+\theta(0)=0,\qquad \theta(\beta)=0 \tag{10}$$

A general form of (9)-(10) has been studied in [3, 5-10] in the form of

$$\begin{cases} y''(x)=-f(x,y(x)), & 0<x<1 \\ \alpha y(0)-\beta y'(0)=0, & \gamma y(1)+\delta y'(1)=0 \end{cases} \tag{11}$$

where $f(x,y)$ is continuous and positive in $(0,1)\times(0,+\infty)$ and $\lim\limits_{y\to 0}f(x,y)=+\infty$. It has been shown that this problem under appropriate conditions on $f(x,y)$ has a unique positive solution.

3 Solutions and discussions

Equations (9)-(10) were solved numerically for its special cases of $k(s)=s^m$ (m =1.0 to 5.0), $h(s)=s$, $\alpha=-2.0$ to -0.1, β =1.0, $n=$ 0.5 to 2.5, by utilizing the shooting technique. The numerical results are presented in the following figures. It may be seen that for each fixed $m>0$, and $n>0$, the problem has exactly one positive solution.

In **figures 1-3**, the numerical results are presented for $k(s)=s^3$, $n=$ 0.5 1.0, 1.5, and 2.5, the diffusion distribution $\theta(s)$ with each α (= –0.1, –0.5, and –2.0) respectively. Let $\sigma=\theta(0)$, it is clearly that for each fixed α, the $\theta(s)$ increases with the decreasing of n, the physical meaning of this is that the diffusion distribution decreases with power law parameter n, and this behavior is qualitatively true with the cases of $k(s)=s$ and $k(s)=s^5$ (**figures 4-7**). **Figure 8** describes the behavior of $\theta(s)$ with the changes of parameter α, from the figure we note that $\theta(s)$ decreases sharply with the increasing of α.

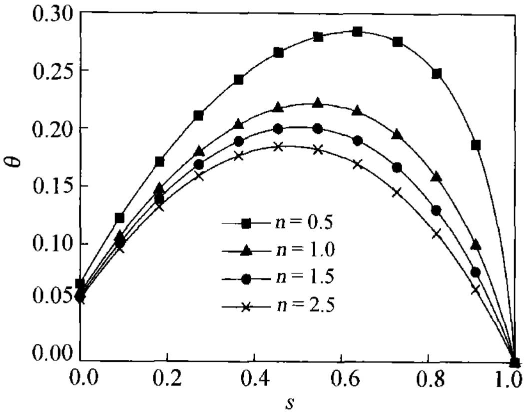

Figure 1 Flux distribution for α=–0.1, β=1.0, $k(s)=s^3$.

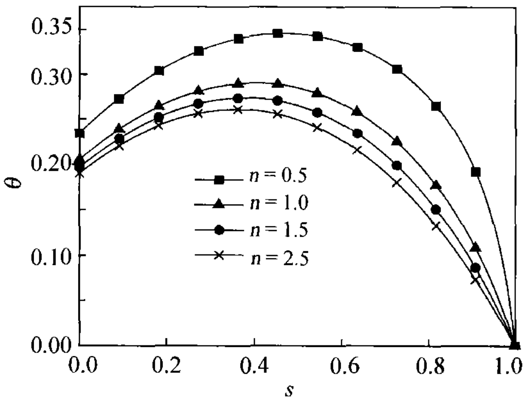

Figure 2 Flux distribution for α=–0.5, β=1.0, $k(s)=s^3$.

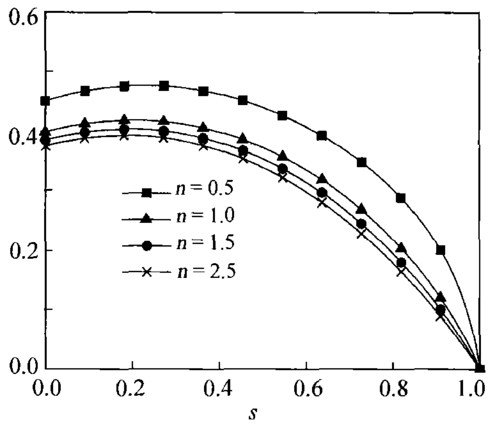

Figure 3 Flux distribution for α=–2, β=1.0, $k(s)=s^3$.

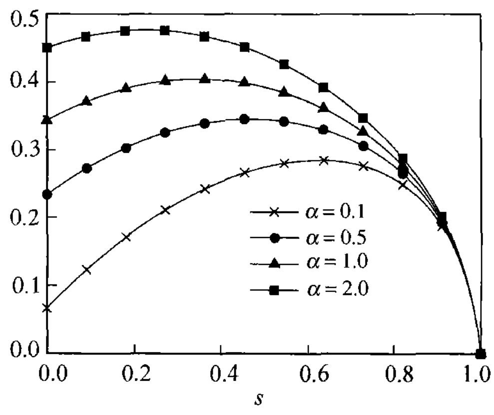

Figure 4 Flux distribution for β=1.0, $k(s)=s^3$, n=0.5 to 2.5.

Figures 9-10 show for each fixed α and each fixed n, the behavior of $\theta(s)$ with changes of $k(s)=s^m$ (m =1.0, 3.0, and 5.0), it is evident that $\theta(s)$ increases with the decreasing of m.

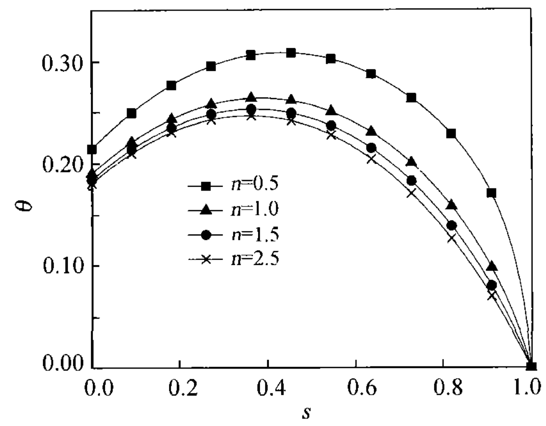

Figure 5 Flux distribution for α=–0.5, β=1.0, $k(s)=s^5$.

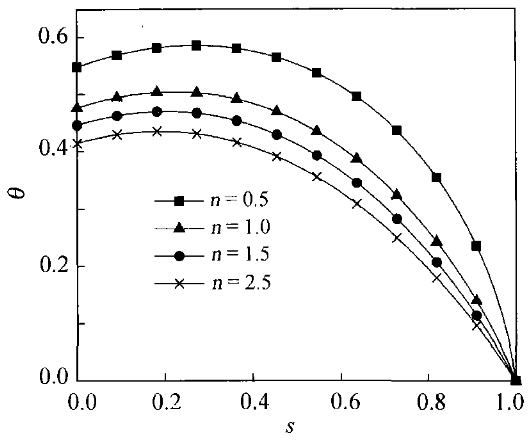

Figure 6 Flux distribution for α=–2, β=1.0, $k(s)=s$.

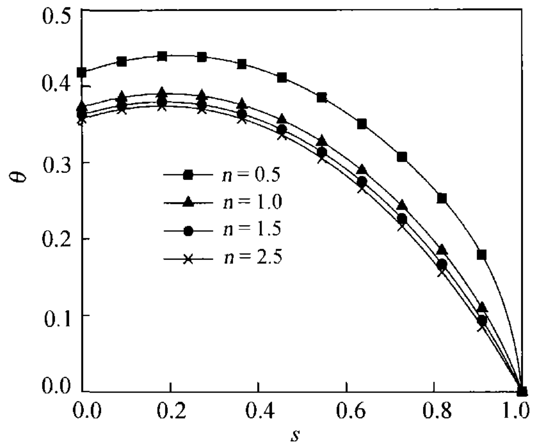

Figure 7 Flux distribution for α=–2, β=1.0, $k(s)=s^5$.

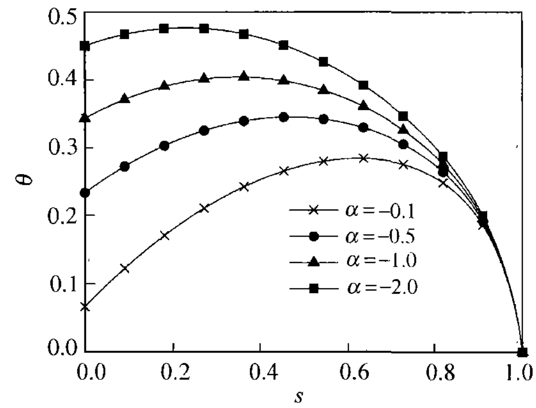

Figure 8 Flux distribution for $k(s)=s^3$, n=0.5, β=1.0.

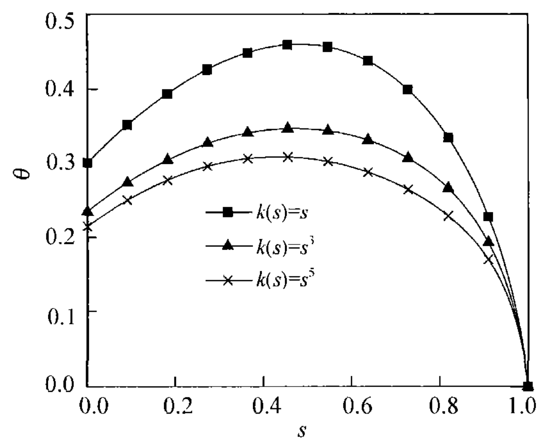

Figure 9 Flux distribution for α=–2, β=1.0, n=0.5 to 2.5.

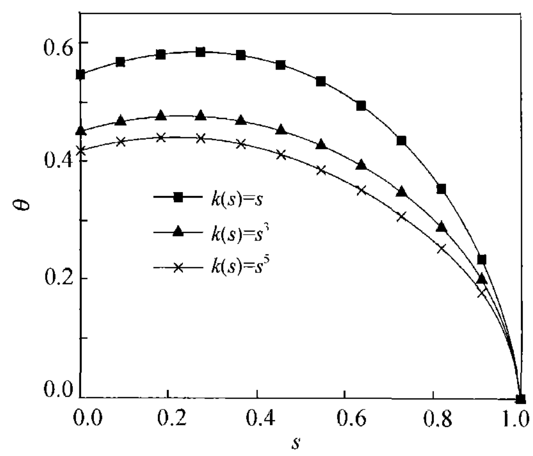

Figure 10 Flux distribution for α=–2, β=1.0, n=0.5 to 2.5.

4 Conclusions

(1) A free boundary value problem for generalized diffusion equation was studied and the numerical solution was presented for $k(s)=s^m$ (m =1.0 to 5.0), $h(s)=s$, and parameters α (α = –2.0 to –0.1), β =1.0, n (n = 0.5 to 2.5), by utilizing the similarity transformation and shooting technique.

(2) For specified $\alpha(\alpha<0)$ and $k(s)=s^m(m>0)$, the diffusion flux function $\theta(s)$ decreases with power law parameter n.

(3) For each fixed $\alpha(\alpha<0)$, the flux function $\theta(s)$ decreases with the increasing of $m(m>0)$.

And for each specified $k(s) = s^m (m > 0)$, flux $\theta(s)$ decreases sharply with the increasing of α.

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