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Automation

Analysis and confirmation of fixed points in logistic mapping digital-flow chaos

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Abstract: The fixed points in logistic mapping digital-flow chaos strange attractor are studied in detail. When $k=n$ in logistic equation, there exist no more than $2n$ fixed points, which are deduced and proved theoretically. Three corollaries about the fixed points of logistic mapping are proposed and proved respectively. These theorem and corollaries provide a theoretical basis for choosing parameter of chaotic sequences in chaotic secure communication and chaotic digital watermarking. And they are testified by simulation.

Key words: chaos; logistic mapping; fixed point; strange attractor

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1 Introduction

Reference [1] discovered infinite fixed points in the logistic mapping in researching logistic mapping digital-flow chaos and describing their strange attractors. Clearly, confirming these fixed points has an important significance for creating chaotic sequences. Because chaotic sequences have been applied to chaotic secure communication [2] and chaotic digital watermarking [3], it is very important to analyse and ascertain these fixed points.

Based on the research of reference [1], this paper firstly analyses the fixed points in chaotic regions of logistic mapping, and secondly proves that there exist no more than 2^n fixed points theoretically when $k=n$ in logistic equation, and finally proposes three corollaries on the fixed points.

2 Strange attractor of logistic mapping digital-flow chaos [4,5]

Logistic equation comes from a model of population dynamics [6]. A simple one-dimension logistic mapping is defined by the following difference equation.

$$x_{k+1} = \mu x_k (1 - x_k) \quad (1)$$

where $x \in (0, 1)$, $\mu \in (1, 4)$.

Countless discrete points aperiodically locate on a protruding parabola and shape its strange attractor, which have been discussed in detail in references [1,2,7,8].

Attractor is the end-result of system action, which behaves in points or set in the phase space. There are always three kinds of attractors in dissipation system, *i.e.* constant attractor, period attractor and chaotic attractor [9]. Constant attractor reflects the damp movement when x state (or y) directly attenuates or surgely attenuates with the value of time t, such as steady crunode and focus in two-dimension spaces; Period attractor behaves the period movement, such as limit cycle; Chaotic attractor, another attractor differing from the above ones, is one that has numeric dimensions in phase space, such as the strange attractor like 'helmet' in logistic mapping [1,7,8].

Definition If $f^k(x_0) = f \cdots f(x_0) = f^{(k-1)}(x_0)$ then x_0 is said to be a fixed point with the k^{th} degree.

3 Analysis and confirmation of fixed points in logistic mapping

With the increase of μ, digital-flow chaotic output of logistic mapping experiences an evolution from fixed points, period-doubling divarication (double times, four times...), fitful chaos to chaotic state.

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Whether are there some fixed points which can not create chaos in chaotic regions? Reference [1] has proved that there are infinite fixed points in logistic mapping if $\mu\in[3.571448, 4]$, and there appear at least two fixed points for each μ. However, this paper discusses the logistic equation in detail, and discovers that there are other fixed points for each μ which are also ascertained. How are these infinite fixed points in chaotic regions confirmed? For this purpose, a theorem and three corollaries to answer the question should be proposed. And they are verified by simulation.

Theorem For logistic mapping $x_{k+1}=\mu x_k(1-x_k)$, $x_k\in(0, 1)$, $\mu\in[3.571448, 4]$, there exist infinite fixed points, when k=n, existing no more than 2^n fixed points as follows.

$$\begin{cases} x_{n,0}^{(1)}=1/2+1/2\sqrt{1-4/\mu*x_{n-1,0}^{(1)}} \\ x_{n,0}^{(2)}=1/2-1/2\sqrt{1-4/\mu*x_{n-1,0}^{(1)}} \\ x_{n,0}^{(3)}=1/2+1/2\sqrt{1-4/\mu*x_{n-1,0}^{(2)}} \\ x_{n,0}^{(4)}=1/2-1/2\sqrt{1-4/\mu*x_{n-1,0}^{(2)}} \\ \vdots \\ x_{n,0}^{(2^n-1)}=1/2+1/2\sqrt{1-4/\mu*x_{n-1,0}^{(2^{n-1})}} \\ x_{n,0}^{(2^n)}=1/2-1/2\sqrt{1-4/\mu*x_{n-1,0}^{(2^{n-1})}} \end{cases} \tag{2}$$

and $x_{0,0}=(\mu-1)/\mu$ (3)

Where $x_{n,m}^{(h)}$ is the h^{th} fixed point while logistic mapping has the n^{th} degree, m is rank of logistic equation in processing.

Proof Logistic mapping $x_{k+1}=\mu x_k(1-x_k)$ is a discrete equation, so the induction for prove is used.

(1) When k=0, equation (1) is given by

$$x_1=\mu x_0(1-x_0) \tag{4}$$

From the definition of fixed point, if x_1=x_0, x_0 is a fixed point, then $x_{0,0}=(\mu-1)/\mu$. Hence equation (3) holds.

(2) When k=1, equation (1) has the form

$$x_2=\mu x_1(1-x_1) \tag{5}$$

If $x_2=x_1$ then $x_{1,1}=(\mu-1)/\mu=x_{0,0}$, substituting x_1 in equation (4) into x_{11}, obtain

$$\begin{cases} x_{1,0}^{(1)}=1/2+1/2\sqrt{1-4/\mu*x_{0,0}} \\ x_{1,0}^{(2)}=1/2-1/2\sqrt{1-4/\mu*x_{0,0}} \end{cases}.$$

Therefore equation (2) is true.

(3) When k=2, equation (1) is represented by

$$x_3=\mu x_2(1-x_2) \tag{6}$$

If $x_3=x_2$ then $x_{2,2}=(\mu-1)/\mu=x_{0,0}$, substituting x_2 in equation (5) into x_{22}, obtain $x_1^2-x_1+x_{0,0}/\mu=0$ then $x_{2,1}^{(1)}=x_{1,0}^{(1)}$, substituting x_1 in equation (4) into $x_{2,1}^{(1)}$, obtain

$$\begin{cases} x_{2,0}^{(1)}=1/2+1/2\sqrt{1-4/\mu*x_{1,0}^{(1)}} \\ x_{2,0}^{(2)}=1/2-1/2\sqrt{1-4/\mu*x_{1,0}^{(1)}} \end{cases}.$$

$x_{2,1}^{(2)}=x_{1,0}^{(2)}$, substituting x_1 in equation (4) into $x_{2,1}^{(2)}$, obtain

$$\begin{cases} x_{2,0}^{(3)}=1/2+1/2\sqrt{1-4/\mu*x_{1,0}^{(2)}} \\ x_{2,0}^{(4)}=1/2-1/2\sqrt{1-4/\mu*x_{1,0}^{(2)}} \end{cases},$$

Therefore equation (2) is true.

(4) When k=n−1, equation (1) is given by

$$x_n=\mu x_{n-1}(1-x_{n-1}) \tag{7}$$

If $x_n=x_{n-1}, \Rightarrow x_{n-1,n-1}=(\mu-1)/\mu=x_{0,0}$

$$\Rightarrow x_{n-1,n-2}^{(1)}=x_{1,0}^{(1)} \Rightarrow \begin{cases} x_{n-1,n-3}^{(1)}=x_{2,0}^{(1)} \\ x_{n-1,n-3}^{(2)}=x_{2,0}^{(2)} \end{cases} \Rightarrow$$

$$\Rightarrow x_{n-1,n-2}^{(2)}=x_{1,0}^{(2)} \Rightarrow \begin{cases} x_{n-1,n-3}^{(3)}=x_{2,0}^{(3)} \\ x_{n-1,n-3}^{(4)}=x_{2,0}^{(4)} \end{cases} \Rightarrow$$

$$\ldots\Rightarrow \begin{cases} x_{n-1,0}^{(1)}=1/2+1/2\sqrt{1-4/\mu*x_{n-2,0}^{(1)}} \\ x_{n-1,0}^{(2)}=1/2-1/2\sqrt{1-4/\mu*x_{n-2,0}^{(1)}} \end{cases}$$

$$\vdots$$

$$\ldots\Rightarrow \begin{cases} x_{n-1,0}^{(2^{n-1}-1)}=1/2+1/2\sqrt{1-4/\mu*x_{n-2,0}^{(2^{n-2})}} \\ x_{n-1,0}^{(2^{n-1})}=1/2-1/2\sqrt{1-4/\mu*x_{n-2,0}^{(2^{n-2})}} \end{cases}.$$

Hence equation (2) holds.

(5) Considering k=n, equation (1) has the form

$$x_{n+1}=\mu x_n(1-x_n) \tag{8}$$

If $x_{n+1}=x_n$, then $x_{n,n}=(\mu-1)/\mu=x_{0,0}$

$$\Rightarrow x_{n,n-1}^{(1)}=x_{1,0}^{(1)} \Rightarrow\ldots\Rightarrow \begin{cases} x_{n,1}^{(1)}=x_{n-1,0}^{(1)} \\ x_{n,1}^{(2)}=x_{n-1,0}^{(2)} \end{cases} \Rightarrow$$

$$\vdots$$

$$\Rightarrow x_{n,n-1}^{(2)}=x_{1,0}^{(2)} \Rightarrow\ldots\Rightarrow \begin{cases} x_{n,1}^{(2^{n-1}-1)}=x_{n-1,0}^{(2^{n-1}-1)} \\ x_{n,1}^{(2^{n-1})}=x_{n-1,0}^{(2^{n-1})} \end{cases} \Rightarrow$$

$$\cdots \Rightarrow \begin{cases} x_{n,0}^{(1)} = 1/2 + 1/2\sqrt{1-4/\mu * x_{n-1,0}^{(1)}} \\ x_{n,0}^{(2)} = 1/2 - 1/2\sqrt{1-4/\mu * x_{n-1,0}^{(1)}} \end{cases}$$

$$\vdots$$

$$\cdots \Rightarrow \begin{cases} x_{n,0}^{(2^n-1)} = 1/2 + 1/2\sqrt{1-4/\mu * x_{n-1,0}^{(2^{n-1})}} \\ x_{n,0}^{(2^n)} = 1/2 - 1/2\sqrt{1-4/\mu * x_{n-1,0}^{(2^{n-1})}} \end{cases}.$$

Therefore equation (2) is true.

From what have been proven above, if k equals to any integer, equation (2) is all true. Obviously, it can be concluded that there exist 2^n fixed points from equation (2).

What have been proven above is based on the condition of square roots being real numbers. In view of some square roots having complex numbers, these fixed points do not exist, so when $k=n$, there exist no more than 2^n fixed points.

Corollary 1 For logistic mapping there have $x_{k+1} = \mu x_k(1-x_k)$, $x_k \in (0,1)$, $\mu \in [3.571448, 4]$, when $k=n$, there exist no more than 2^{n-1} pairs of fixed points.

Proof From the theorem, it can be easily concluded that $x_{n,0}^{(1)}$ and $x_{n,0}^{(2)}$, $x_{n,0}^{(3)}$ and $x_{n,0}^{(4)}$, …, $x_{n,0}^{(2^n-1)}$ and $x_{n,0}^{(2^n)}$ are 2^{n-1} pairs of fixed points. In view of some square roots that may be complex numbers, these fixed points do not exist. Therefore there exist no more than 2^{n-1} pairs of fixed points.

When n=1, equation (2) is given by:

$$\begin{cases} x_{1,0}^{(1)} = 1/2 + 1/2\sqrt{1-4/\mu * x_{0,0}} \\ x_{1,0}^{(2)} = 1/2 - 1/2\sqrt{1-4/\mu * x_{0,0}} \end{cases}.$$

If $x_{0,0} = (\mu-1)/\mu$ is substituted into above equation, then $x_{1,0}^{(1)} = (\mu-1)/\mu$, and $x_{1,0}^{(2)} = 1/\mu$, which have been proven in theorem 1 in reference [1].

Corollary 2 For logistic mapping there have $x_{k+1} = \mu x_k(1-x_k)$, $x_k \in (0,\ 1)$, $\mu \in [3.571448,\ 4]$, at $k=n$, the number of fixed points becomes more and more with the increase of μ. When μ=4, there exist 2^n fixed points at most.

Proof A qualitative analysis is only made. From the formal formula:

$$x_{n,0} = 1/2 \pm 1/2\sqrt{1-4/\mu * x_{n-1,0}}.$$

where $\mu \in [3.571448, 4]$, obtain:

(1) When μ <4, then $4/\mu$ >1, then $4\,x_{n-1,0}/\mu > x_{n-1,0}$. With the increase of iterating steps, $4\,x_{n-1,0}/\mu$ is larger than 1 quickly. Then $1-4\,x_{n-1,0}/\mu < 0$, square roots exit complex numbers, that is, these fixed points do not exist.

(2) When μ=4, the formal formula is $x_{n,0} = 1/2 \pm 1/2\sqrt{1-x_{n-1,0}}$. Because $x_{n-1,0}$ is always less than 1, at μ=4, logistic equation has 2^n fixed points at most.

(3) With the increase of μ, μ is nearer to 4, so $4\,x_{n-1,0}/\mu$ is not easily larger than 1, and square roots of complex numbers are fewer, and the number of fixed point is more.

For example when μ=3.6.

$x_{0,0} = 0.72222222$.

$x_{1,0}^{(1)} = 0.72222222$, $x_{1,0}^{(2)} = 0.27777777$.

$x_{2,0}^{(1)} = 0.72222222$, $x_{2,0}^{(2)} = 0.27777777$.

$x_{2,0}^{(3)} = 0.91573971$, $x_{2,0}^{(4)} = 0.08426029$.

$\vdots$

$x_{3,0}^{(5)} = 1/2 + 1/2 * \sqrt{-0.01748856}$ is a complex number.

$x_{3,0}^{(6)}$ is a complex number too.

Therefore these two fixed points do not exist.

Corollary 3 For logistic mapping there have $x_{k+1} = \mu x_k(1-x_k)$, $x_k \in (0,\ 1)$, $\mu \in [3.571448,\ 4]$, at $k=n$, each fixed point converges to a same value ,

$x_{0,0} = (\mu-1)/\mu$.

Simulation, shown in **figure 1**, indicates that each fixed point converges to the same value, $x_{0,0}$. But it is unstable since $|f'(x_{0,0})| > 1$. Figure 1(a) is the output of logistic mapping in fixed point, where k =50, μ = 3.6, x_0=0.9157397099. Figure 2(b) is output of logistic mapping in fixed point, where k=50, μ=4, x_0=0.93301270189

Figure 2 is output and strange attractor of logistic mapping in fixed point, where k =100, μ =4, x_0=0.75.

Figure 3 is distributing chart of fixed points in logistic mapping, where k = 5, μ = 4, x_0 = 0.75. From this graph, these fixed points can obviously avoided when the chaotic mapping is built.

Practically, because of the limit of computer precision, it is very difficult to get the accurate fixed points. When the computer iterates n steps, the system all enters chaotic state. This is the reason why there are infinite fixed points in theory, but in practice, it is too difficult to find them except for some real number without arithmetical compliment (such as 0.25, 0.75, see figure 2). Figure 1 shows the conclusion.

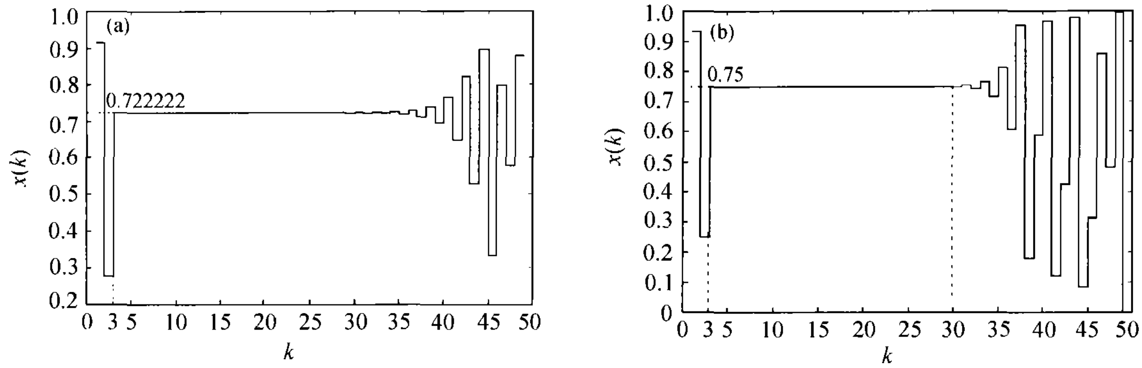

Figure 1 Output of logistic mapping in fixed point, (a) k=50, μ=3.6, x_0=0.9157397099; (b) k=50, μ=4, x_0=0.93301270189.

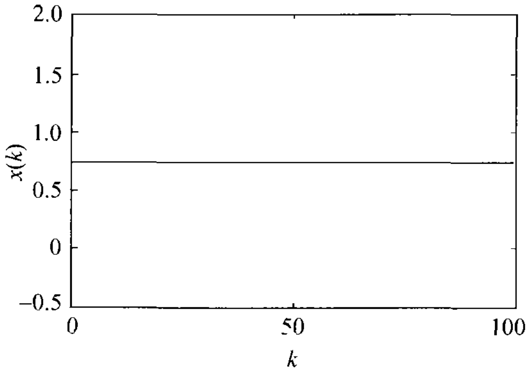

Figure 2 Output and strange attractor of logistic mapping in fixed point, k=100, μ=4, x_0=0.75.

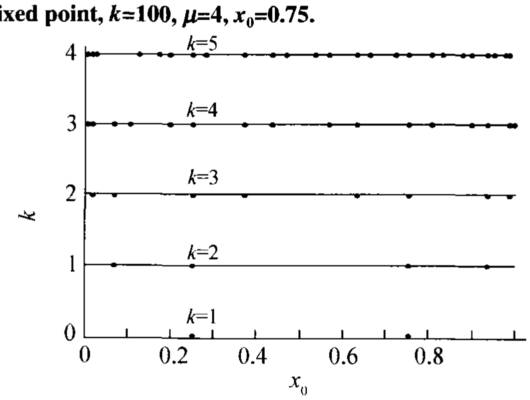

Figure 3 Distributing chart of fixed points in logistic mapping.

4 Conclusions

In this paper, the fixed points of logistic mapping in chaotic regions are analyzed. One theorem and three corollaries are presented to determine all possible fixed point with k^{th} degree, and give their expressions. These theorem and corollaries provide a theoretical basis for choosing the chaotic sequence parameters in chaotic secure communication and chaotic digital watermarking. Fortunately, it is not easy to enter these points for logistic mapping. Practically there are two reasons as follows: (1) The fixed point $x_{0,0}=(\mu-1)/\mu$ is a repelling fixed point for $\mu\in$ [3.571448, 4] because of $|f'(x_{0,0})|>1$; (2) Computer has limited precision. However, the results of theory and simulation imply that using logistic mapping as secure communication might not be safe (see figures 1-3).

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