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Analytical solutions to a compressible boundary layer problem with heat transfer

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Abstract: The problem of momentum and heat transfer in a compressible boundary layer behind a thin expansion wave was solved by the application of the similarity transformation and the shooting technique. Utilizing the analytical expression of a two-point boundary value problem for momentum transfer, the energy boundary layer solution was represented as a function of the dimensionless velocity, and as the parameters of the Prandtl number, the velocity ratio, and the temperature ratio.

Key words: compressible boundary layer; momentum and heat transfer; analytical solution

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1 Introduction

When an expansion wave advances into a stationary fluid bounded by a wall, a boundary layer flow is established along the wall behind the wave. This type of problems had been solved numerically by Mires [1] and are often important in the studying of phenomena involving non-stationary waves (*e.g.*, shock tube studies, initiation of detonation studies, *etc.*). For the sake of simplicity, we restrict ourselves to the considerations of perfect gas, thus it will be assumed that μ (coefficient of viscosity), κ (thermal conductivity) are proportional to T and that C_p (specific heat capacity at constant pressure) and Pr (Prandtal number, $\mu C_p/\kappa$) are independent of T (temperature). Consider a plane laminar flow with spatial coordinates (x, y), corresponding velocity components (u, v) and $\mathrm{d}p/\mathrm{d}x = 0$, for steady flow, the boundary layer equations for $x > 0$ can be written as [1-3]:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

$$P = \rho RT \tag{4}$$

The boundary conditions are:

$$u(x,0) = u_w, \; u(x,\infty) = u_e \tag{5}$$

$$v(x,0) = 0 \tag{6}$$

$$T(x,0) = T_w, \quad T(x,\infty) = T_e \tag{7}$$

where w is the conditions at wall; e is the flow external to the boundary layer.

2 Similarity transformation

2.1 Stream function and similarity variable

Introduce the stream function ψ, and the similarity variable η by the expressions:

$$\psi = \sqrt{2u_e x v_w}\, f(\eta), \quad \eta = \sqrt{\frac{u_e}{2xv_w}} \int_0^Y \frac{T_w}{T(x,y)}\mathrm{d}y .$$

Substituting (8) into (1)-(7) yields the following boundary value problems:

$$f'''(\eta) + f(\eta)f''(\eta) = 0, \quad 0 < \eta < +\infty \tag{9}$$

$$f(0) = 0, f'(0) = \xi, f'(+\infty) = 1 \tag{10}$$

$$\bar{T}''(\eta) + Pr\, f(\eta)\bar{T}''(\eta) = -\frac{Pr u_e^2}{C_p T_e}(f''(\eta))^2 ,$$

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$$0<\eta<+\infty \tag{11}$$

$$\overline{T}''(0)=\lambda,\ \overline{T}(+\infty)=1 \tag{12}$$

where the prime denotes differentiation with respect to η, $\xi=f'(0)=u_w/u_e$ is the velocity ratio parameter ($\xi=0$ corresponding to the classical Blasius problem), $\lambda=T_w/T_e$ is the temperature ration parameter, and $0<\xi<1$ for an expand wave.

In this study, we demonstrate that equations (9)-(12) may be further reduced by inducing a suitable transformation, as result, the analytical solutions for the problems are obtained.

2.2 Two-point boundary value problem

Define a similarity variable transformation to both momentum and energy equations as Zheng and Zhang [5-8]:

$g(t)=f''(\eta)$ (dimensionless shear force)

$t=f'(\eta)$ (dimensionless velocity, $0\le\xi\le t\le 1$) (13)

$w(t)=\overline{T}(\eta)$ (dimensionless heat distribution)

Substituting equation (13) into (9)-(12), noted that $f''(\eta)>0, 0<\eta<+\infty$ and $f''(+\infty)=0$, yields the following singular nonlinear two-point boundary value problems:

$$g(t)g''(t)+t=0,\qquad 0\le\xi<t<1 \tag{14}$$

$$g'(\xi)=0,\qquad g(1)=0 \tag{15}$$

$$w''(t)+(1-Pr)w'(t)g'(t)/g=-\frac{Pru_e^2}{C_pT_e},\quad 0\le\xi<t<1 \tag{16}$$

$$w(\xi)=\lambda,\qquad w(1)=1 \tag{17}$$

It may be seen from the derivation process that only the positive solutions of the problems are physically of significance. Clearly, the singular nonlinear two-point boundary problems (14)-(15) are de-coupled and can be solved first, and the solution of equations (16)-(17) may be established by utilizing the solution of equations (14)-(15).

3 Solutions of boundary value problems

3.1 Solution of boundary value problems (14)-(15)

As far as the positive solution of equations (14)-(15) are concerned, the analytical solutions of $g(t)$ in $[\xi, 1]$ may be established by utilizing Bernstein's theorem and Tauberian's theorem. Recently, Zheng *et al.* [5-8] discussed some general cases of power law fluid boundary layer equations for $0<n\le 1$ and some general nonlinear boundary value problems corresponding to the surface moving in the direction or opposite to the direction of the stream. Sufficient conditions for existence, non-uniqueness, uniqueness and analytical positive solutions to the problem were obtained utilizing the perturbation and shooting techniques.

Let $\alpha=g(\xi)$ $(=f''(0))$ (skin friction coefficient), then $g'(\xi)=0$, $g''(\xi)=-\frac{\xi}{\alpha},\cdots$, and

$$g^{(n+3)}=-g^{-1}\left[\sum_{p=0}^{n}C_n^p g^{(p)}g^{(n+2-p)}+\sum_{p=1}^{n+1}C_n^p g^{(p)}g^{(n+3-p)}\right].$$

By use of induction, all $g^{(n)}(\xi)\,(n=1,2,\cdots)$ may be established. It can be seen that each derivative of $g(t)$, of fourth or higher order, can be expressed in terms of those of lower order, thus all derivatives of $g(t)$ depend only on the first three, and therefore on the skin friction α, *i.e.*

$$g(t)=\alpha-\frac{\xi}{2\alpha}(t-\xi)^2-\frac{\xi}{6\alpha}(t-\xi^2)^3-\frac{\xi^2}{24\alpha^3}(t-\xi^4)-\cdots \tag{18}$$

In terms of $g(1)=0$, we can obtain

$$\alpha=\sum_{n=1}^{\infty}\frac{|g^{(n)}(\xi)|}{n!}(1-\xi)^n \tag{19}$$

Since $g^{(n)}(\xi)\le 0$, formula (19) yields

$$\alpha=\frac{\xi}{2\alpha}(1-\xi)^2+\frac{\xi}{6\alpha}(1-\xi^2)^3+\frac{\xi^2}{24\alpha^3}(1-\xi^4)+\cdots \tag{20}$$

Theorem: Let $0\le\xi<1$, then

$\sqrt{\tfrac{1}{6}[1-\xi^2(3-2\xi)]}<\alpha<(1-\xi)\sqrt{\tfrac{1}{2}(1+\xi)}$.

Proof. Since $g(t)$ is positive, and from the concavity of $g(t)$ in $[\xi, 1]$, we have

$$\frac{\alpha}{1-\xi}(1-t)<g(t)\le\alpha,\ \text{for}\ \xi<t<1 \tag{21}$$

From equations (14)-(15), we obtain

$$0=\alpha-\int_{\xi}^{1}\frac{s(1-s)}{g(s)}\mathrm{d}s \tag{22}$$

The lower and upper bounds of $g(t)$ used from (22) in (23) give the proof.

The theorem may be used to estimate the value of wall friction for specified ξ. In particular, when $\xi=0$, we obtained immediately

$0.4082\approx\sqrt{\tfrac{1}{6}}<\alpha<\sqrt{\tfrac{1}{2}}\approx 0.7071$.

The singular nonlinear two-point boundary problems (14)-(15) are solved by utilizing the shooting technique. The numerical results are shown in **figure 1**, which indicates that the skin friction $\alpha = g(\xi)$ decreases with the increasing of the velocity ratio parameter ξ, and for each fixed ξ, the shear force $g(t)$ decreases with the increasing of the tangent velocity t in $[\xi, 1]$. In particular, the case $\xi = 0$ is the well-known Blasius problem for flowing past a semi-infinite flat plate and is included herein for the sake of completeness.

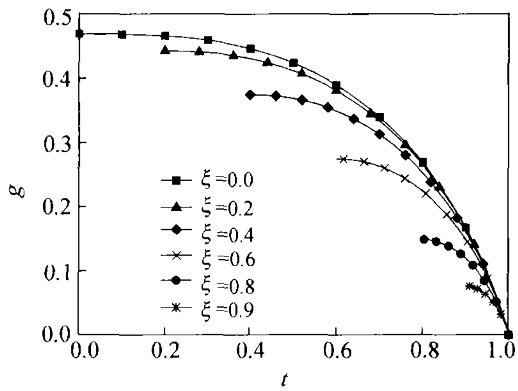

Figure 1 Shear force distribution for ξ=0.0 to 0.9.

3.2 Solutions of two-point boundary value problems (16)-(17)

Utilizing the analytical solution of equations (14)-(15), the boundary layer energy equations (16)-(17) were solved and the analytical solution was represented as:

$$w(t) = -\frac{Pr \cdot u_e^2}{C_p T_e}\int_\xi^t (g(s))^{1-Pr} ds \int_\xi^t (g(s))^{Pr-1} ds + \frac{1-\lambda-\frac{Pr \cdot u_e^2}{C_p T_e}\int_\xi^1 (g(s))^{1-Pr}(\int_\xi^s (g(z))^{Pr-1} dz) ds}{\int_\xi^1 g^{Pr-1}(s) ds} \times$$
$$\int_\xi^t (g(s))^{Pr-1} ds + \frac{Pr \cdot u_e^2}{C_p T_e}\int_\xi^1 (g(s))^{1-Pr} ds \times$$
$$\int_\xi^1 (g(s))^{Pr-1} ds + \frac{Pr \cdot u_e^2}{C_p T_e}\int_\xi^t (g(s))^{1-Pr} \times$$
$$(\int_\xi^s (g(z))^{Pr-1} dz) ds + \lambda \tag{23}$$

Solution (23) may be reduced to a simple form and it may be seen that the solution is only affected by parameters Pr, ξ and λ. In particular, when $Pr = 1$, a very simple form is given by

$$w(t) = -\frac{u_e^2}{2C_p T_e}(t-\xi)^2 + \frac{1-\lambda+\frac{u_e{}^2}{2C_p T_e}(1-\xi)^2}{1-\xi}(t-\xi)+\lambda \tag{24}$$

The results reveal the relation between the momentum and energy transfer as well as the effects of the velocity ratio ξ and temperature ratio λ.

4 Conclusions

(1) The problem of momentum and heat transfer in a compressible boundary layer behind a thin expansion wave was solved by the application of the similarity transformation and the shooting technique. The analytical solutions for the problem have been presented.

(2) Utilizing the analytical expression of the two-point boundary value problem for momentum transfer, the energy boundary layer solution was represented as a function of the dimensionless velocity, and as the parameters of the Prandtl number, the velocity ratio, and the temperature ratio.

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